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ABSTRACT

This research-oriented document is designed to give teachers insights into many of the causes of instructional problems in mathematics and to enable teachers to plan instruction which will be more responsive to identified needs. Individual chapter authors synthesize mathematics research findings, which provide three functions: (1) to describe what has been; (2) to find out what exists; and (3) to find out what is possible. Chapters in many cases have "counter-point" responses from well-known researchers in mathematics education. Topics covered include: (1) The Value of Mathematics Education Research; (2) Curriculum; (3) National Assessment; (4) Children's Thinking; (5) Teacher's Decision Making; (6) Process-Product Research; (7) The Sex Factor; (8) Problem Solving; (9) Computers; and (10) Calculators. (MP)

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# Mathematics Education Research: Implications for the 80's

Elizabeth Fennema, Editor

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# Foreword

In these days characterized by great emphasis on the so-called basic skills, it is important that we look at where we have been, where we are today, and where it is possible to go in mathematics instruction.

This research-oriented publication is to serve just such purpose. Not that these research reports and their interpretations will give teachers prescriptions for teaching students, but they do give insights into many of the causes of instructional problems in mathematics, thus enabling teachers to plan instruction more responsive to identified needs.

The authors of these chapters have synthesized mathematics research findings for busy teachers. Their realistic observations cover both sexes' feelings about mathematics, the problem-solving processes learners most often use, and the characteristics of interactions between teachers and pupils in the instructional process. For instance, Cooney reports on some interesting research showing that the warm, caring teacher has a positive effect in low socioeconomic status classrooms, but often a negative effect in high SES classes. Nevertheless, overall research findings indicate the effectiveness of a supportive teacher.

The summaries by DeVault and Weaver relative to the uses of computers and calculators, respectively, are particularly timely. Not many educators have dealt effectively with the technology explosion to ensure its contribution to the delivery of instruction.

Dr. Izaak Wirszup, following his comparative study of science and mathematics education in Russia and America, recommended sweeping changes in U.S. education, involving mathematics instruction throughout the K-12 sequence, and in both undergraduate and graduate curriculums.[1]

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[1] Izaak Wirszup, "The Soviet Challenge," *Educational Leadership* 38 (February 1981): 358-360.

Neither he nor the authors of this book have reported research regarding the effectiveness of the use of textbooks in mathematics instruction. Since a large number of teachers rely on texts for direction, it is reasonable to expect that they are an important variable in the quality of mathematics instruction.

Research as interestingly presented as in this booklet goes a long way toward dispelling myths and stereotypes too often retained by most of us. Our theories and beliefs are revised or changed in the light of new perceptions and experiences. We hope that this much-requested publication will serve to enlighten us and motivate each of us to commit ourselves to recast mathematics education, as we take leadership in developing the curriculums that will educate the citizenry of the new century.

LUCILLE G. JORDAN
*President, 1981-82*
*Association for Supervision and Curriculum Development*

# Introduction:
## The Value of Mathematics Education Research

Elizabeth Fennema

A president of the National Council of Teachers of Mathematics (NCTM) once said he had never learned anything from research that was of use in teaching. He also quoted a nonreferenced source as saying that "Evaluation in a great number of countries shows that educational research and research into the field of the pedagogy of mathematics have virtually no influence on school practice" (Egsgard, 1978, p. 554). While some mathematics educators share this strongly negative opinion, others feel just the opposite. Begle and Gibb (another NCTM president) wrote that "research in mathematics education generates an improvement in the teaching of mathematics, and direction for the development of mathematics curricula . . . Research in mathematics education has provided bricks for edifices of cognitive development, skill learning, concept and principle learning, problem solving, individual difference, attitudes, curriculum, instruction, teaching, and teacher education . . . Every mathematics educator (teacher, and teacher of teachers of mathematics) can benefit from these edifices" (Begle and Gibb, 1980, p. 15).

Egsgard, Begle, and Gibb are (or were) highly respected and active participants in the teaching of mathematics. How could they come to such widely divergent beliefs, and which, if either, describes the contribution of research to mathematics education? In all honesty, I believe that neither of these extreme positions is valid. The first underestimates what research can contribute while the other overestimates what is possible at this time.

This chapter, along with the other chapters in this book, is an attempt to realistically show how knowledge of research can contribute to improvement in mathematics education.

Mathematics education research can make contributions to the teaching of mathematics in at least three areas: (1) description of what has been, (2) description of what is, and (3) description of what is possible. In addition, by building theories, research aids in putting the world of mathematics education in broad perspective.

One important contribution of educational research is *to describe what has been,* and to trace the influence of the past on current educational practice. While using different methodologies than experimental or status research, historical study is an important type of educational research. Although it often indicates that we have failed to learn from previous experience, it provides part of the knowledge necessary to make important curricular decisions. Dessart's chapter on curriculum includes research of this type.

Another important role of research is *to find out what exists.* Research provides systematic description of specific situations to see what an objective examination of reality reveals. Surveys, ethnographic, and observational studies are examples of this type of research. Results from such status studies are often surprising and in conflict with widely held beliefs. For example, many people believe that children today are not learning computational skills. The results from the National Assessment of Educational Progress (see Carpenter and others in this book) give information about changes in and the status of children's computational skills. These results indicate that on a nationwide basis, children are doing quite well in computational activities. While many teachers believe they accurately know how they interact with the learners in their classrooms, some studies indicate that teachers are not totally aware of their interaction patterns. What actually occurs in teacher-pupil interactions is reported in the teaching chapter by Grouws and Good. Some processes learners use to solve problems are reported in the problem solving chapter by Kantowski. The chapter on sex-related differences reports girls' and boys' feelings about mathematics.

One specific type of status research is that which deals with *evaluation.* Such studies can serve a variety of purposes. Formative evaluation is a continuous process enabling changes to be made in instructional programs. Summative evaluation determines whether learning has occurred and can be used for a range of programs, from the specific to those that are statewide or nationwide. The chapter by Carpenter and others is an interpretive report of a summative evaluation.

Status research studies are extremely valuable to all who are concerned with mathematics education. They help us do away with faulty thinking based on myths and inaccurate perceptions of reality. They help us know what is possible, and how changes can be made in instruction in order to achieve certain goals.

Another contribution of education research is *to find out what is possible*. Traditional experimental research, which involves precise design and analyses, is carried out in this type of study. For example, conditions of instruction are manipulated in fairly well-defined ways to see if improved learning by specifically described groups of learners follows. The chapters by DeVault and Weaver describe studies in which computers or hand-held calculators were used in a variety of ways in teaching mathematics. Many of these studies were experimental, involving direct comparison of learning by students involved in different types of instruction.

Human beings like to understand what goes on in the world and to put that perceived world into some kind of order. When educational researchers attempt to put an order on what they see happening in relation to education, they are *building theories*, another contribution of educational research. These theories attempt to explain why something has happened and to predict future events. Sometimes they have direct implications for mathematics classrooms. However, many times while a theory might help in understanding the variables at work in a classroom, it has no direct implications for instruction. Perhaps the best known theory that many believed had direct implications for the planning of curriculum is that of cognitive development explicated by Piaget. Hiebert's chapter discusses his perceptions of how at least one portion of this theory has not been particularly helpful in planning mathematics curriculum. He talks about how conservation is apparently a component of some major mathematical ideas, but does not appear to be essential in the learning of these ideas. The chapter by Cooney is an attempt to build a theory about teacher behavior.

Missing from this list of contributions of mathematics education research is any mention of providing information that will tell a mathematics teacher, at any level, what to do in her or his classroom. This is a deliberate omission because I firmly believe research cannot give precise direction to what a specific teacher should do in a particular classroom. This is not to say that research is not helpful to classroom teachers. It is only that research cannot, nor should it even if it could, tell teachers exactly what they should be doing as they plan, conduct, and evaluate instruction.

Research is but one of many ways knowledge is gained by educators (DeVault, in press). Scholarly writing is another way. No one would ever doubt that John Dewey has had a major impact on classrooms and certainly his writing would not be described as research. Teachers also can and must depend on their own experience for knowledge. The wisdom of the classroom teacher should not be negated.

What, then, can research contribute to mathematics education? Research can give us new insight into solution of old and new problems; it can suggest improved classroom procedures; it can make us more objective in our perceptions; and it can make us more thoughtful in all ways as we go about the teaching of mathematics. Research cannot tell us what should be done. Only values determine questions of this type. Research alone cannot determine what a teacher should do on Monday, or on any day of the week. But thoughtful educators will continue to use results of research as one mechanism for the improvement of education for all people.

In the past, many have held unrealistically high expectations of the kind of knowledge research can provide. Researchers may have inadvertently contributed to these high expectations by how they reported and interpreted specific studies. The purpose of this book is to put implications from research in a realistic framework which also considers knowledge gained from other sources.

This book is an attempt to put research in such a form that findings can be readily assimilated by practitioners. The authors were selected because they had worked for a long time in the area in which they were writing. Their task was not to report on individual studies but to synthesize research findings and put them in the context of scholarly writings and classroom teacher wisdom. Each author reported that the task was difficult, but the results reflect their commitment to making research available to nonresearchers in a form that can be used to improve the learning of mathematics by all.

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# I. Curriculum

Donald J. Dessart

In the late 1940s, a professor at the University of Wisconsin complained that it was far easier to move a cemetery in mid-January that it was to change the curriculum. Although such a statement may have been true of the curriculum in mathematics before 1950, it certainly is not true of the activity related to the mathematics curriculum after 1950. Since that time, a veritable revolution has taken place.

In this chapter we will review events in research related to this revolution that hold lessons for the curriculum of the future. The next 30 years will offer challenges related to many problems—energy, inflation, declining economics, and international problems of an explosive nature. The role of the mathematics curriculum in this maelstrom will provide demands of monumental proportions for teachers, researchers, and other professionals in mathematics education.

The terms "curriculum" and "mathematics curriculum" are widely used and subject to many different interpretations. *For our purposes, the mathematics curriculum will mean the mathematical content that is to be mastered by the learner and the instructional policies and procedures that are used to organize this content with the intention of promoting effective learning.*

## Mathematical Content

The selection of mathematical content for instruction in the schools is subject to many demands and pressures. These demands fall into three categories: psychological, sociological, and structural. In an ideal situation, they play equal and complementary roles in the curricular process;

but in reality there is a tendency to stress one demand over another, depending on the pressures of society.

**Psychological Demands**

From time to time, the selection of mathematical content appears to fall under the influence of two broad psychological theories: (a) the behavioristic, mechanistic theories that view the learner as mastering pieces of mathematical content that collectively produce a whole of learning; and (b) the holistic or field theories that view the learner as comprehending the entirety or gestalt of learning, usually through insight or sudden inspiration.

If we subscribe to the behavioristic point of view, then the goal of instruction is to subdivide the mathematical content into objectives or bits of mathematical knowledge that must be mastered in a "best" sequential order for the learner to succeed. The recent emphasis on behavioral objectives is an example of the results of such an influence.

On the other hand, if we subscribe to the holistic or field theories, then we attempt to provide the fabric that pervades learning and provides a cognitive structure. The more recent emphasis on problem solving as a broad task of understanding the problem, developing a plan and reconsidering the plan, fits into that theory. In the problem-solving process, emphasis is on instilling in the learner the tolerance for manipulating the pieces of the problem in an orderly but not necessarily sequential manner so the learner may arrive at an insight or solution to the problem.

Most teachers subscribe to an eclectic point of view, believing that certain kinds of mathematical content, such as addition or subtraction algorithms, should be organized with a behavioristic theory as the guiding influence; whereas other mathematical content, such as in understanding or finding a geometrical proof, are guided by a holistic theory. In teaching an algorithm, it is important to identify the sequential steps of the process and to have students master these steps. In teaching a geometrical proof, it is more important to provide a varied selection of viewpoints for examining the problem, with the goal that perhaps one of these viewpoints will lead to a solution.

Most teachers are comfortable with a selective point of view that allows one or the other theory to dominate, depending on the content to be learned. But there are "purists" who simply feel that *all* mathematical learning can be described by behavioral objectives or mechanistic descriptions; and there are other purists who feel that *all* mathematical learning should be mastered by a random or incidental organization that leads to insight and solution.

Research studies conducted during the 1930s and 40s attempted to test the effectiveness of these two theories. For example, many studies compared drill instruction with meaningful instruction. In drill instruction the goal was to subdivide the learning experience into parts in which drill or practice was provided for each subdivision. In meaningful instruction an attempt was made to provide a fabric of understanding—a varied set of experiences that would provide a variety of insights into the mathematical content to be mastered. At first it seemed that drill and meaningful instruction were antithetical, but later it became apparent that the two were complementary and that, normally, drill should be preceded by meaningful instruction.

### Sociological Demands

Sociological demands for curriculum organization stress individuals' needs for certain mathematical content in order to serve themselves and society. A student must learn arithmetic, algebra, and geometry to satisfy certain vocational demands such as those required of an electrician, an accountant, or an engineer. Consequently, the selection of content must recognize and satisfy these needs. Psychological theories involve academic questions that teachers and curriculum workers can address in the laboratory, while sociological demands are more diverse to identify and to provide for in the curriculum.

There have been various attempts to identify sociological needs. The Commission on Post-War Plans of the National Council of Teachers of Mathematics, meeting after World War II, pointed out the inadequacies of the mathematics curriculum and identified 29 mathematical competencies that should form the minimal or core curriculum. (For instance: can measuring devices be used? Are students skillful in the use of interest tables, income tax tables? Can they construct scale drawings? and so on.) As one reviews these 29 competencies, it is clear that they are socially oriented.

Curriculum makers are alert to the sociological needs of students, and teachers certainly feel the demands, particularly from students who ask, "What is this good for?" Such a question may more truly indicate psychological problems, rather than sociological problems, with the curriculum. Nevertheless, providing for the needs of society is a valid demand placed on teachers and curriculum makers.

### Structural Demands

The structure of the discipline of mathematics is often the primary criterion guiding the organization of the curriculum. Certainly, during the modern mathematics movement in the United States, the structure of mathematics was the major force influencing the design of the curriculum. Most modern mathematics projects were concerned with structural questions such as: What is a number? What is the difference between a number and a numeral? What are the laws of arithmetic; such as, the commutative law, the associative law, and so on? What is a variable? What is a function? What is an equation? What is a group? What is a field? All of these questions were aimed at understanding the structure of the discipline.

### Interplay of Psychological, Sociological, and Structural Demands

From time to time, one of the three demands seems to influence curriculum makers in mathematics to a greater extent than the others. During times of economic pressures, as during the Great Depression, mathematics teachers lean toward sociological demands as a means of justifying the mathematics curriculum. At other times, such as during the modern mathematics movement, the structural demands seem to be the major factor in designing the curriculum, and yet at other times the psychological demands seem more prevalent, for example, during the recent competency and accountability movements. Obviously, teachers and curriculum makers must attempt to satisfy the current demands of society; but, on the other hand, we would hope that the curriculum would be of a nature that it could satisfy all of the needs of society. Research in this direction or efforts to find suitable models of curriculum design have not been highly successful in the past. Obviously, the complexities of such a model are mind-boggling, and we can readily appreciate that viable models will not be developed quickly or easily. In the meantime, teachers and curriculum makers must be aware of the psychological, sociological, and structural demands, and attempt to provide an amalgamation of the three whenever possible.

## Instructional Procedures

The organization of the curriculum depends on the instructional procedures that are to be employed in the classroom. Obviously, the curriculum used in a classroom in which the teacher plays a highly directive role is quite different from a curriculum in which the teacher plays a less directive role. Consequently, curricular organization can be viewed as a function of teacher intervention in the classroom. Three kinds of teacher control will be considered: (a) lecturing or underlining, (b) guided learning, and (c) pure student learning.

### Lecturing or Underlining

In lecturing or underlining, the teacher is exerting maximum control of the class activities and the sequencing of events in the classroom. A program designed for this kind of teacher intervention provides a structure or outline of topics to be covered, but leaves the manner of presentation and class activities largely to the discretion of the teacher. Problem sets and exercises are also provided, but the selection of these are left to the discretion of the teacher. This pattern of curricular organization is probably the most predominant pattern used in American schools and colleges and is undoubtedly viewed as the most efficient and practical curricular organization.

### Guided Learning

Guided learning may play a significant role in a classroom in which the teacher attempts to engage students in the learning activities. It may also play a large role in programmed instruction or various kinds of individualized instruction. In these procedures, the curriculum must play a more definitive guiding role than in a class in which the teacher lectures. The sequence of events and the manner of presentation must be developed by the curriculum maker rather than solely by the teacher.

Although guided learning is used less frequently in schools than standard lecturing, it does have a more significant role in elementary than in secondary schools. There was a period in the late 1950s and 60s when it appeared that programmed learning would make inroads in the American classroom. But its vitality was short-lived and gave way to more conventional teaching procedures.

### Pure Student Learning

The school in which the student develops a curriculum according to his or her own needs is Utopian. Very few elementary or secondary schools employ such procedures, although a few college classes, primarily at the graduate level, have used pure learning methods. In such a procedure, the teacher's role is one of counselor and provider of initial directions. The curriculum, as such, is highly flexible, undeveloped to a large extent prior to the class meeting. Individual freedom and creativity are highly emphasized, and a teacher of extremely mature and learned judgment is needed to function in such a structure.

## The Evolving Mathematics Curriculum

### The Elementary School Curriculum

Prior to the 1950s, the elementary school curriculum was concerned primarily with developing computational skills with whole numbers, fractions, and decimals. Thorndike dominated much of the thinking. His view was summarized in *The Psychology of Arithmetic* (1924): "We now understand that learning is essentially the formation of connections or bonds between situations and responses . . . and that habit rules in the realm of thought as truly and as fully in the realm of action" (p. vi). The elementary school curriculum under Thorndike's influence consisted largely of identifying specific stimuli (for example, $3 + 4 =$) and specific responses (7) for the skills of computation. Attempts to provide more than developing the "bonds" or "connections" for specific stimuli and their responses were regarded as not only superfluous but a detraction from the main goal of learning.

Reactions to this theory came from mathematics educators like Brownell and Chazal (1935) who wrote an exposition of the drill theory. Brownell, Chazal, and others of this time advocated a "meaning theory" which emphasized the inner relationships of the number system as well as the social uses of arithmetic. Several decades later the curriculum developers of the School Mathematics Study Group (SMSG) relied on the theories of Brownell for justification of their interpretation of "meaning" as understanding the structure of mathematical systems.

Searches for meaning in the elementary school curriculum led naturally to the consideration of other topics. Geometry, which was primarily a domain of the secondary schools, began to make its appearance in the elementary schools largely through the efforts of the modern mathematics innovators. The geometry of the elementary school dealt primarily with informal understandings of concepts to be taught with more precision in later years. For example, points, lines, rays, half-planes, angles, triangles were treated informally in anticipation of more formal definitions in the high school curriculum.

The modern mathematics movement of the 1950s and 1960s emphasized "meaning" also, but primarily from a content point of view. Emphasis was placed on understanding non-decimal numeration systems, the laws of number systems, and the more formalized rules of arithmetic. Drill of computational skills was emphasized to a far lesser degree than in former years and the goal of instruction was to develop an understanding of the number system and its properties. This philosophy held that such formal understandings would make students more flexible and able to apply mathematics to a variety of situations.

The enchantment with modern mathematics began to wane when mathematicians and educators criticized the emphasis on formalism as being too artificial and not consistent with usage in modern life. The decline of students' computational skills was blamed on the modern mathematics curriculums, and the emphasis on skill development under the guise of "basic skills" made its reappearance during the 1970s. By now, the meaning theory was still popular but was coupled with drill as a viable means of developing computational skills.

The basic skills movement led to a concern that today's children were able to compute better than children of former years but were not capable of applying these skills to solving problems. A reaction to this concern was expressed by the NCTM, which recommended that problem solving should be the focus of the curriculum of the 1980s and that basic skills should be broadened to encompass more than merely computational facility.

### The Middle and Junior High School Curriculum

Prior to the modern mathematics movement, the middle grades (six, seven, eight) were devoted largely to consolidating computational skills learned in earlier years as well as broadly emphasizing social applications. Students of these years were exposed to applications of arithmetic in banking, installment plan buying, homemaking, and other social uses. This emphasis prompted modern mathematics proponents to regard the junior high school years as a "wasteland" of mathematics education. Beberman was reported to have stated that children of these years were more interested in "the number of angels that danced on the head of a pin" than in social applications.

This opinion motivated modern mathematics curriculum developers to replace social applications with topics of a more highly abstract mathematical nature. The formalism begun in the elementary curriculum was extended to the junior high school years. Understanding equation solving and "proofs" for arithmetic rules were stressed. Greater emphasis on informal geometry and the introduction of new topics, such as probability and statistics, were promoted. During these years, programmed instruction, individualized instruction, and mathematical laboratories were also recommended for the middle years. Many a junior high school housed a special classroom-laboratory where children could pursue special projects related to paper folding, string art, games, and other special topics. One sometimes wonders if these laboratories were actually an "escape" from the more formal material of the modern mathematics programs.

Two events during the 1970s modified the curriculum of those years. The hand-held calculator was becoming more readily available and the metric system was being promoted. Although some teachers deplored the calculator as detracting from students' acquisition of computational skills, the National Advisory Committee on Mathematical Education recommended that a hand-held calculator be made available to each student by the end of the eighth grade. The need for the nation to implement the metric system was recognized, and the junior high school years were seen as an appropriate time to introduce students to this system.

### The High School Curriculum

Teaching algebra in grade nine to 15-year-olds had been traditional for many, many years dating back to the late 1800s. Although algebra became somewhat more formal during the modern mathematics movement, its nature did not change radically. Traditional teachers who had emphasized skill development prior to the modern mathematics years were able to continue this emphasis with the modern textbooks by merely ignoring the text material between problem sets.

On the other hand, the geometry curriculum was subject to greater change. Birkoff devised a set of postulates for geometry, making greater use of those properties of the real number system which were unavailable to Euclid. These postulates, published in the *Annals of Mathematics* of April 1932, became the basis for a "new" geometry textbook by Birkhoff and Beatley (1940) of the inauspicious title, *Basic Geometry*.

The influence of Birkhoff's work and that of Hilbert on the SMSG approach to geometry is clearly shown by Brumfiel (1973). In 1980, the SMSG approach dominated the secondary school geometry curriculum in the United States. It is one of the few innovations of SMSG that has avoided most successfully the criticisms leveled at other modern mathematics programs. The major contribution of the SMSG approach was to introduce postulates concerning abstract rulers and protractors into high school geometry. These postulates, which capitalize on properties of the real number system, filled the logical gaps of the Euclidean geometry previously taught in high school.

Although this approach to geometry is the predominant one, it is certainly not the only geometry curriculum used in the United States. NCTM's 36th yearbook, *Geometry in the Mathematics Curriculum* (Henderson, 1973), identifies no less than seven approaches to formal geometry in the senior high school. These approaches include the conventional synthetic Euclidean geometry, approaches using coordinates, a transformational approach, an affine approach, a vector approach, and an approach to satisfy all teachers: an eclectic program in geometry.

Before the 1950s, the high school curriculum beyond geometry usually consisted of a second year of algebra followed by a semester of solid geometry and a semester of trigonometry. This program has been replaced by a unified approach that includes topics from traditional advanced algebra, theory of equations, linear algebra, probability and statistics, and the calculus. The calculus as a separate subject is studied by relatively few students; in 1977-78, only 4 percent of the 17-year-olds in the United States elected it (National Science Foundation, 1980).

## Evaluation of Programs

As the modern mathematics programs made inroads into the schools of the United States, administrators, teachers, and parents wanted to know which of the programs—the modern or the traditional—was better for their children. Early attempts at such evaluations might be termed macro-evaluations since they made broad comparisons of classes using traditional curriculums and those using modern programs.

### Minnesota National Laboratory Evaluations

In the spring of 1964, the Minnesota National Laboratory reported receiving a grant of $249,000 from the National Science Foundation The general purpose of the grant was to study the effectiveness of various kinds of mathematics courses, including both conventional courses and new courses developed by the School Mathematics Study Group, the University of Illinois Committee on School Mathematics, the Ball State Teachers College, and the University of Maryland Mathematics Project.

The study was designed to determine differences in achievement between pupils instructed with conventional materials and those instructed with one of the modern programs. Volunteer teachers from a five-state area were invited to participate for a two-year period, teaching a class with conventional materials during the first year, and two classes, one with conventional and one with modern materials, during the second year. Students were tested to determine initial measures of achievement, final achievement, and retention.

In 1968 the results of the evaluations (Rosenbloom and Ryan, 1968) did not show that the modern programs had strongly increased or decreased achievements. There were some exceptions to the overall findings, but even those did not strongly favor the modern curriculums. In subsequent investigations on attitudes, interests, and perceptions of proficiency, no unequivocal differences were found. Dessart and Frandsen (1973, p. 1178) commented on these results:

A conclusion which might have been drawn from these projects was that the experimental materials tested were not worth the vast resources that had gone into their development. The project reporters suggested that such a conclusion should be tempered because of possible lack of validity of the achievement tests for measuring significant objectives of the experimental programs or because of possible lost impact due to poor teacher performance with experimental materials.

## New Hampshire Studies

Studies of a smaller scope were also conducted to ascertain advantages or disadvantages of the modern programs. Typical of these was a study performed in New Hampshire which compared groups of students studying from modern, transitional, and traditional materials during 1963-1967 (Austin and Prevost, 1972). In 1965, the pupils studying the modern materials outdistanced the other two groups on the Otis Mental Abilities Test but performed lower on computation testing. By 1967, the modern group scored higher on computation, concepts, and applications tests. Throughout all of these testings, students' abilities to perform arithmetic computations declined.

A pattern was emerging from these evaluations: students in the modern programs seemed to achieve somewhat higher in comprehension of mathematical concept measures but scored lower in measures of computational ability than their counterparts in conventional programs. This led to the humorous comment attributed to Beberman that a modern schoolboy knows that the sum of two natural numbers is a unique natural number, but he doesn't know which one!

## National Longitudinal Study of Mathematical Abilities (NLSMA)

The NLSMA was a five-year study that attempted to compare the effectiveness of conventional and modern textbooks as well as the effects of student attitudes and backgrounds of teachers (Begle and Wilson, 1970). Three populations designated X, Y, and Z were studied in the fall and spring of each year. The X population, fourth-grade students in 1962, and the Y population, seventh-graders, were tested for the full five years. The Z population, consisting of tenth-graders, was examined during a three-year period.

The data assembled were enormous. The results related to textbooks indicated that: (1) the variability of means associated with textbook groups decreased as grade levels increased; (2) the SMSG textbook groups performed better than conventional groups on comprehension, analysis, and application levels, but not on the computational level; and (3) some of the modern textbooks produced poor results on all levels from analysis through computation.

The NLSMA revealed that comparing textbooks is at best exceedingly difficult and complex, leading to very few clear generalizations. Many variables must be examined to provide a broad view of the comparisons; otherwise, one may be in the position of comparing textbooks on a single criterion which may not be significant. This lesson was costly to learn as evaluation projects, such as NLSMA, enjoyed the era of federal funding that may never be matched in the future.

## Curricular Variable Identification and Study

As we have seen from the evaluations of programs that were conducted during the 1960s and 1970s, the results were not clear-cut and generalizations were very difficult to make. The major obstacle to arriving at meaningful evaluations was the large number of variables that needed consideration. Merely identifying these variables was not sufficient because, in most cases, the variables were not well understood and were not easily measured. During the 1980s, more intensive efforts in variable identification and study seem to be a natural consequence of the research of the previous two decades.

### Adjunct Questions and Teaching Problem Solving

One of the difficulties faced by nearly every mathematics teacher is that of teaching "word" problems. An often used but questionable instructional procedure consists of the teacher presenting a model problem to the class and leaving the model on the chalkboard as a pattern for working other problems. Successful students learn to select numbers from the textbook problems to fit the chalkboard model and perform similar operations. Doing enough problems of this kind, students memorize a pattern or algorithm for solving a particular problem. Later, when they meet this type of problem again, it is hoped they will apply the same pattern or algorithm.

Although this instructional procedure has some merit, it probably suffers from two weaknesses: (1) the student is not really "thinking through" the problem but rather is engaging in a "matching-analysis" procedure; and (2) the pattern or algorithm may be forgotten by the student when he or she meets this particular problem in isolation of other similar problems. This instructional procedure does, however, represent an attempt to guide the student through a problem-solving process. Methods of motivating or leading students to engage in problem solving are needed by teachers at nearly all levels. One method that may hold promise is to guide the students by key questions designed to highlight or emphasize the essential elements of a problem and encourage the student to consider these elements. Little research has been done with the usefulness of such questions in mathematics education, but research in other areas may provide some clues for us.

For example, Rothkopf (1966) reported a research study with 159 college students who studied a written passage and were given questions either before or after reading the material. Rothkopf concluded that questions given after reading the material, called "adjunct questions," have both specific and general facilitative effects on post-reading performance.

Mathematics educators have given insufficient attention to the possible uses of adjunct questions—that is, questions that would follow a written paragraph and would be designed to focus the attention of the student on the salient features of that paragraph. For example, if word problems were followed with specific, written questions that would focus the attention of students on the elements of the problem and thus force them to "think through" the problem, would we be more successful in motivating students?

It seems clear that during the 1980s efforts to improve the teaching of word problems will be a priority if the disturbing evidence of the latest NAEP results concerning the success of our students in dealing with word problems influences the directions of teaching (see Chapter II). In such efforts, the usefulness of adjunct questions seems to be an attractive possibility. Furthermore, such questions could be designed to accommodate the Polya (1945) model of understanding the problem, devising a plan, carrying out the plan, and looking back at the problem.

A serious question that arises in the use of adjunct questions is their value in facilitating the questioning abilities of students; that is, will students exposed to adjunct questions initiate questions of their own in attacking word problems or, even further, in a given cultural situation, would students be able to formulate (identify) the problem to be solved? We can hope that such would be the case, but there is a danger that students would become far too dependent on the adjunct questions. Research could provide further insights.

### Advance Organizers

Experienced teachers know that before beginning a new topic with a class they must provide an introduction. This introduction probably consists of two phases: (a) a review in which the teacher helps students recall relevant facts, concepts, and principles; and (b) an overview in which the teacher attempts to provide students with general and overarching insights concerning the new material to be studied. This bit of conventional wisdom was given a more precise and theoretical foundation through the work of Ausubel (1968) and others working in the area of advance organizers.

Ausubel pointed out that advance organizers facilitated meaningful learning in three ways: (a) they mobilize relevant anchoring concepts already established in the learner's cognitive structure; (b) at an appropriate level of inclusiveness they provide optional anchorage which promotes initial learning and resistance to later loss, and (c) they render unnecessary much rote memorization students often resort to because they lack sufficient numbers of key-anchoring ideas. Ausubel (1968, p. 148) succinctly summarized the characteristics of advance organizers: "In short, the principal function of the organizer is to bridge the gap between what the learner already knows and what he needs to know before he can successfully learn the task at hand."

Since a good introduction is important to a learning task, it would seem that research on advance organizers should be a productive area for improving teaching. It appears that a relevant question is not whether advance organizers should or should not be used but rather in what ways can advance organizers be constructed to be most effective. Consequently, it seems cogent that any future work on advance organizers should concentrate on their useful characteristics rather than on a testing of their use or disuse.

One of the central issues related to advance organizers is the "goodness of fit" among the elements of the advance organizer, the new material to be learned, and the characteristics of the cognitive structure of the student. The first step in establishing a "good fit" is a matter of identifying the central mathematical concepts and principles of the new material and providing for generalizations of these in the organizer. As difficult as this may be, it is not an insurmountable task. The next step, that of designing the organizer to complement the existing cognitive structure of the student, is an extremely complex task. What is an effective organizer for one student may be utter confusion for another because of the differences in their cognitive structures.

Work by Bloom and his associates is related to the question of designing an organizer to complement the cognitive structure of the student. Bloom (1980, p. 383) identifies "cognitive entry characteristics" as the specific knowledge, abilities, or skills that are necessary prerequisites for a particular learning task. He points out that such prerequisites correlate .70 or greater with measures of achievement of the task.

The critical problem, then, is one of designing effective advance organizers or providing for appropriate cognitive entry characteristics to fit the cognitive structures of the student. But isn't this the crucial problem in teaching? The solution to this problem necessitates studies designed to ascertain the nature of the existing cognitive structure of the student and providing for the further development of these structures; an exercise similar to the one a conscientious teacher employs in tutoring a student over a learning difficulty. The initial step is usually one of attempting to determine the nature of the existing knowledge of the student before beginning the remediation or, in this case, the design of the advance organizer.

### Behavioral Objectives

The use of behavioral objectives made a significant contribution to mathematics education because it sharpened the focus upon describing the desired student behavior at the conclusion of a learning experience. As with advanced organizers, the use of behavioral objectives made "pedagogical sense" and they became popular. Research related to the use of behavioral objectives for the most part supported them. As Begle (1979) observed, in over 30 studies completed to determine if the use of behavioral objectives could lead to greater student achievement, half of these studies indicated that student achievement was improved or speeded up. In perhaps only one case did the use of behavioral objectives have a negative effect. In each study the question was essentially one of testing whether students who were exposed to behavioral objectives before a learning experience would achieve higher than those who were not so exposed. The evidence seemed to support their use.

In spite of a promising beginning, the interest in behavioral objectives seemed to decline. As with most "bandwagon" movements, teachers are led initially into the movement by advocates who preach a panacea is at hand. Such was the case with the behavioral objective movement; but when teachers were required to take hours from their busy teaching days not only to write lengthy behavioral objectives for the mathematical topics of the curriculum, but also to monitor their attainment with a class, fatigue soon took its toll. What had initially been regarded as a panacea became a terrific demand on teachers' time.

It soon became evident that the higher cognitive goals of instruction—critical thinking, creating, problem solving—were elusive of capture in the behavioral objective mold. Many of the advocates of the movement recognized this deficiency but felt that, given sufficient time and energy, these goals could be described by behavioral objectives. After all, if we really understand the behavior we desire, then surely we should be capable of describing it.

## Learning Hierarchies

Closely related to the notions of advance organizers and behavioral objectives is that of the learning hierarchy. Whereas the advance organizer intends to provide an introduction to the learning experience, and the behavioral objective focuses on the final behavior exhibited by the student at the conclusion of a learning experience, the learning hierarchy is a means of organizing learning tasks to achieve a final goal-objective. The process requires describing the mathematical objective and asking what the student should be able to do in order to achieve that objective. Answering such a question raises the necessity of defining subtasks that should be achieved and subtasks of these subtasks so that a network of tasks is generated. In this network, achieving the final task is dependent on achieving all of the subtasks.

Much of the research conducted with learning hierarchies has been in the form of validation studies. Phillips and Kane (1973) constructed seven different orderings of 11 subtasks for rational number addition. A test was designed to assess mastery at each of the 11 subtasks of the hierarchy. One hundred forty-two students were assigned randomly to the seven treatments corresponding to the seven different orderings of the hierarchy. No particular sequence was found to be consistently superior on achievement, transfer, retention, and time to complete the sequence.

## Some Lessons Learned from Research on Advance Organizers, Objectives, and Learning Hierarchies

As we contemplate the efforts that have been expended in research related to advance organizers, behavioral objectives, and learning hierarchies, we quite naturally ask, "What have we learned?" The lessons have been modest, but valuable.

First, one of the most valuable lessons is that there is an organization of the objects of the curriculum called its "psychological organization"; that is, an organization that arises from an analysis and study of the mathematical understandings of children. This organization may be quite different from its axiomatic, logical, mathematical organization. The implication of this lesson is that teachers must understand the psychological organization of the curriculum as well as its mathematical organization. The latter has had the benefit of centuries of the axiomatic method since the time of Euclid, whereas the psychological organization has only been given serious consideration in the last 50 to 100 years. One intuitively feels that all of the psychological research findings related to learning mathematics is waiting for a Euclid to organize them into a viable theory of mathematical learning.

Second, we have learned that there is a danger of attempting to fit all mathematical learning into one model which may turn out to be an inadequate model. The use of the behavioral objective model is an example of this inadequacy. Some view such a shortcoming with alarm when, in fact, it is the natural evolution in any theory; that is, when the theory is inadequate, it must be revised.

Third, it is undoubtedly true that good teachers have known from conventional wisdom the necessity of including in their instructional techniques the basic ideas embodied in advance organizers, behavioral objectives, and learning hierarchies. Researchers have made the valuable contribution of organizing this wisdom into carefully devised theories. Although these theories are often incomplete and inadequate, they do represent a serious beginning of a theory of mathematical learning.

## Meaningful Instruction and Drill

Van Engen (1949) discussed three particular theories of meaning: (1) social meaning in which the child understands the mathematics that he or she can observe and use in social situations, (2) structural meaning in which the mathematics becomes meaningful when the child understands the structure of the subject, and (3) the nihilistic theory of meaning which denies meaning to symbols. Brownell (1945) made a statement which is typical of those often quoted, that is, that "Meaning is to be sought in the structure, the organization, and the inner relationships of the subject itself" (p. 481).

In contrast, we might conclude that rote instruction neglects to emphasize the structure, the organization, and the inner relationships of the subject of mathematics and puts emphasis on repetition and fixation of concepts, principles, procedures, or algorithms. In considering a definition for the term "drill," we turn to Sueltz (1953), who discussed drill in the following way: ". . . the words 'drill,' 'practice,' and 'recurring experience' are used to indicate those aspects of learning and teaching that possess elements of similarity and sameness which repeat or recur" (p. 192).

In conclusion, it seems clear that since the research of Brownell and Moser (1949) on meaningful versus mechanical learning, and probably earlier, many mathematics educators subscribed to a statement similar to one by Gibb (1975, p. 59):

> The controversy of drill and practice versus understanding (including the use of hands-on and laborabory types of learning experience) is a longstanding, but unnecessary one. I believe that neither can be considered in isolation from the other. A lot of rote drill and practice in the absence of understanding or useful application does little to promote computational

efficiency. Likewise efforts for developing understanding alone are not effective unless they are tempered with drill and practice to build proficiency in computation, in problem solving, and in thinking logically.

Consequently, it seems reasonable to state that nearly all mathematics educators agree that meaningful instruction and drill go hand in hand, with meaningful instruction simply preceding drill or practice, but that both are necessary for an efficient and profitable learning experience. Willoughby (1970, p. 263) echoed a similar sentiment:

> Virtually everyone who consciously addresses himself to the question of whether it is better for a child to understand mathematical concepts or to commit verbalization of the concepts to memory agrees that understanding is important and desirable. During the recent "revolution" in school mathematics, this point of view became so prevalent that, in some instances, textbooks provided substantial material to help children understand a concept, but virtually no practice or drill work to help them become adept at using it. Available evidence suggests that a child can understand without becoming adept in using the particular skill involved (addition or fractions, for example), but if the skill is one in which he ought to become proficient, practice or drill will be needed. On the other hand, if drill is used without understanding, retention does not seem to be as great, and, of course, the learning of a skill involving the same understanding but different sorts of symbols will be more difficult if the understanding has not been developed.

As noted from Willoughby's comments, drill and practice became unpopular during the modern mathematics movement. The disenchantment with drill actually preceded the modern mathematics revolution. Sueltz (1953, p. 192) observed:

> Twenty-five years ago (about 1928), drill was the common method of learning applied to such school subjects as arithmetic, writing, and spelling. Children were required to write a word 50 times to learn to spell it and the present generation of middle-aged people spent countless minutes in winding up ovals in one direction and then unwinding them in the opposite direction in order to train the muscles to follow the sweeping curve of penmanship. This was drill; it was carried to extremes and became so sterile that during the 10-year period of approximately 1935 to 1945 drill, as a learning procedure, was frowned upon and ridiculed in many educational circles. However, during the same period it remained the dominant pattern employed by many teachers.

The comment by Sueltz that drill remained the dominant pattern employed by many teachers was borne out by research. Milgram (1969) investigated the ways in which elementary teachers used class time in mathematics. A team of observers making twice-weekly observations of 46 intermediate grade teachers in Pennsylvania found the following use of time: (a) going over previous assignment, 25 percent; (b) oral or written drill, 51 percent; (c) introducing new mathematical concepts or developmental activities, 23 percent; and (d) unrelated interruption, 1 percent.

This pattern of instruction has not changed drastically as can be seen from the words of Fey (1979), who analyzed three studies completed by the National Science Foundation (p. 494):

> Despite the difficulty of knowing what teachers understood by the terms "lecture," "discussion," and "individual assignments," the profile of mathematics classes emerging from the survey data is a pattern in which extensive teacher-directed explanation and questioning is followed by student seatwork on paper and pencil assignments. This pattern has been observed in many other recent studies of classroom activity. . .

Fey cites Welch, one of the NSF investigators, who wrote that the pattern of instruction in all mathematics classes he observed was the same, that is, discussion of the previous assignment, discussion of the new material, followed by seatwork for the students in which they worked the next day's assignment with help from the teacher.

## The Textbook

Since the textbook plays such a highly significant role in the life of the student and the teacher, it is surprising that there hasn't been more research related to its effectiveness. Begle (1979, p. 73) summarized the reason for this dearth of research:

> Any two textbooks differ on so many variables that it would be almost impossible to trace the specific variables which cause a specific difference, and without knowing which variables make a difference, we do not know where to start to improve textbooks.

The textbook in many schools is the mathematics curriculum for that grade level. If a topic does not appear in the textbook, it is not taught. This fact of life was certainly used by the curriculum developers of the 1950s and 60s. They knew that in order to change the curriculum, the most effective and efficient method is by changing the contents of textbooks. Since this state of affairs probably will not change in the future, more effort should be expended in research on textbooks. Millions of dollars each year are spent on textbooks, so one can hardly argue the desirability and practicality of such research.

For example, does color make a difference in the effectiveness of learning by children? Most publishers must have concluded that it does make a difference (in sales, at least) as they use color lavishly. Do illustrations, pictures, type style, or the size of the page make a difference? If not, we might question whether funds to develop these features in books could be diverted to other more productive educational uses.

Research on textbooks appears to be a fruitful area of investigation. Some feel that the number of variables is too large; others, on the other hand, feel that research is manageable. For example, Walbesser (1973, p. 76) offered the opinion that all textbooks should contain "(1) performance descriptions of objectives, (2) data on the acquisition of the behaviors described as objectives, and (3) a statement of where the performance list and data are available." He felt so strongly about this viewpoint that he urged a moratorium on the purchase of all textbooks until the author and/or the publisher provided such information!

Walbesser noted that the selection of textbooks is done in a haphazard manner by most selection committees. They depend very heavily on such criteria as endorsement by authorities, bandwagon effects, high identification quotients, little change from previous textbooks, the use of illustrations, the skill of the sales representatives, and the reputations of the authors. Walbesser presented a plan for selecting textbooks that was highly dependent on the use of behavioral objectives and learning hierarchies. This procedure represents a well-defined and rigorous method and provides a promising direction deserving greater attention.

The values of the textbook will probably remain unchanged in the future. Corporations such as IBM, which is a high-technology company at the forefront of innovations, seem to prefer conventional educational practices. Peter Dean (1980), program manager of IBM's Education Developer Services, cited among identifiable trends that "Most managers and employees perceive conventional stand-up classroom instruction as the only 'true' education. Telling tends to be equated with education." Furthermore, he noted that "Most of the self-study material is printed; a modicum of videotape is used" (p. 317).

It seems reasonable for one to conclude that the textbook will continue to be an important tool in the mathematics classroom. More research on its effectiveness and use is certainly in order for the 1980s; in fact, one might place its research in a high priority category.

## Summary

In isolating and studying carefully the variables of the curriculum, we may gain a better understanding of the influence of curriculum on learning. Adjunct questions, advance organizers, behavioral objectives, and learning hierarchies offer fruitful areas for investigations. While research on the textbook's effectiveness is difficult, its significance as a learning tool demands that it be studied more carefully and researched more fully.

The late A. S. Barr, a well-known researcher at The University of Wisconsin, often remarked that research is very similar to mining in that much shoveling is required before hitting paydirt! This analogy is so appropriate for research on the mathematics curriculum. Much work lies ahead, but the rewards of a better and more effective education for our children is a reward well worth the efforts.

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# II. National Assessment

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In the last two decades there has been considerable unrest in mathematics education. The modern mathematics reforms of the 1960s were followed by the back-to-basics movement of the 1970s, leaving the mathematics curriculum with many characteristics of two decades earlier. As the 1980s begin, new recommendations for school mathematics are being proposed (NCTM, 1980).

The earlier reforms in the mathematics curriculum have been based on some general principles of learning or student achievement. The modern mathematics movement embraced principles of meaningful learning and discovery learning (Bruner, 1960); and the back-to-basics movement was, in part, a reaction to a perceived decline in achievement test scores (Advisory Panel, 1977). Progress in improving mathematics learning, however, is going to require a much more careful analysis of students' learning and achievement than accompanied previous reforms.

One of the best measures of the achievement of American students is provided by the mathematics assessment of the National Assessment of Educational Progress (NAEP). The second NAEP mathematics assessment was conducted during the 1977-1978 school year. Exercises covering a wide range of objectives were administered to a carefully selected national representative sample of over 70,000 students at ages 9, 13, and 17. Item sampling procedures were used so that between 250 and 450 exercises were administered to each age group with approximately 2,400 students responding to each exercise. Consequently, the results provide an accurate sampling of the knowledge of elementary and secondary students over a broad range of objectives. The assessment also has the advantage of providing analysis of performance on specific exercises. Data on each exercise were analyzed separately to provide a description of students' performance on specific tasks.

The objectives that guided the development of exercises for the assessment were selected by panels of mathematicians, mathematics educators, classroom teachers, and interested lay citizens to reflect important goals of the mathematics curriculum. These groups concluded that the mathematics curriculum should be concerned with a broad range of objectives. Accordingly, the assessment focused on five major content areas: (1) numbers and numeration; (2) variables and relationships; (3) geometry (size, shape, and position); (4) measurement; and (5) other topics, which included probability and statistics, and graphs and tables.

Each content area was assessed at four levels: knowledge, skill, understanding, and application. Knowledge level exercises involved recall of facts and definitions. This included such tasks as ordering numbers; recalling basic addition, subtraction, multiplication, and division facts; identifying geometric figures; and identifying basic measurement units. Skill exercises involved various mathematical manipulations including computation with whole numbers, fractions, decimals, and percents. Also included were making measurements, converting measurement units, reading graphs and tables, and manipulating algebraic expressions. Understanding exercises tested students' knowledge of basic underlying principles such as the concept of a unit covering in measurement. These exercises were constructed so that students could not simply apply a routine algorithm. Application exercises required students to use their own knowledge or skills to solve problems. Both routine textbook problems and nonroutine problems were included in this category.

In addition to these cognitive areas, a number of affective variables were assessed, as well as students' self-reports of the types of activities they engage in during mathematics class. Also, a special set of exercises assessed students' ability to use a calculator to solve various kinds of problems.

The results of the assessment have been summarized elsewhere (Carpenter and others, 1980b, 1980c, 1981b; NAEP, 1979a, b, c, d). Going beyond these results, we have identified several areas in which the National Assessment has provided information about students' knowledge of mathematics that relates directly to the NCTM recommendations for the mathematics curriculum of the 80s. These areas are: (1) the need for a broader definition of basic mathematical skills, (2) the importance of students' understanding of mathematical concepts and processes, (3) the importance of problem solving as the focus of the mathematics curriculum, (4) documentation of the continued development of mathematical skills, (5) implications of calculators for teaching computational skills, (6) the need to increase and extend students' enrollment in mathematics courses, and (7) students' perceptions of their involvement in mathematics classroom activities.

Caution must be observed in interpreting the results from the NAEP mathematics assessment, which was not designed to identify causes of student performance. We have frequently extrapolated beyond the data in drawing conclusions; other authors would possibly reach different conclusions. However, the conclusions presented here are generally supported by a wide range of exercises in addition to the illustrative exercises reported.

## A Broader Definition of Basic Skills

The National Council of Teachers of Mathematics (1980) recommendations for school mathematics for the 1980s include the recommendation that basic skills in mathematics should be defined to encompass more than computational facility. Geometry, measurement, probability, and statistics are recognized as important areas of basic skills (National Council of Supervisors of Mathematics, 1978; NCTM, 1980). If students' performance on the second mathematics assessment is a measure of instructional emphasis in the United States, we must conclude that the focus of most mathematics programs is on the development of routine computational skills since students demonstrated a high level of mastery of computational skills, especially those involving whole numbers.

Almost all students demonstrated mastery of basic number facts. About two-thirds of the 9-year-olds could perform simple addition and subtraction computation using algorithms for regrouping. By age 13, almost all students could perform simple computations involving addition, subtraction, and multiplication. Most of the older students were successful with the more difficult calculations such as those summarized in Figure II-1. Students encountered greater difficulty with whole number division and operations with fractions and decimals.

Performance was significantly lower, however, on exercises assessing basic noncomputational skills. In general, the only noncomputational skills for which students demonstrated a high level of mastery were those involving simple intuitive concepts or those concepts or skills they were likely to have encountered and practiced outside of school. This is reflected in students' knowledge of geometric terms. Students were familiar with common everyday terms like square or parallel, but not with terms like tangent and hypotenuse that are used less commonly in everyday vernacular. Over 95 percent of the 13-year-olds could identify squares and parallel lines; but even by age 17, fewer than 60 percent of the students were

**Figure II-1. Whole Number Computation**

| Exercise | Percent Correct | |
|---|---|---|
| | Age 13 | Age 17 |
| a) 4285<br>3273<br>+5125<br>? | 85 | 90 |
| b) Subtract 237 from 504 | 73 | 84 |
| c) 671<br>× 402<br>? | 66 | 77 |

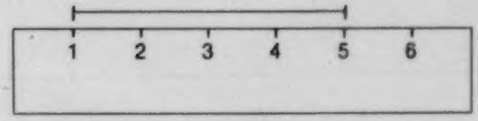
familiar with terms like tangent and hypotenuse. The failure of students to learn basic geometric concepts is illustrated by the fact that only about a fifth of the 13-year-olds and a third of the 17-year-olds could solve a problem involving a simple application of the Pythagorean Theorem.

Performance on measurement exercises generally followed the same pattern as the geometry results. Most students were familiar with measurement concepts and skills that would likely be encountered and practiced outside of school, such as recognizing common units of measure, making simple linear measurements, and telling time. They had a great deal of difficulty, however, with many other basic measurement concepts and skills, especially those involving perimeter, area, and volume. Another basic skill area in which performance was generally low was probability and statistics. Fewer than half the students at any age level demonstrated even a tentative understanding of most basic probability concepts.

## The Importance of Understanding

As results in the previous section showed, students failed to master a broad range of basic skills. Further, many of the skills appear to have been learned at a rote, superficial level. Students' performance showed a lack of understanding of basic concepts and processes in many content areas, such as measurement and computation with fractions. For example, almost all students could make simple linear measurements. Over 80 percent of the 9-year-olds and 90 percent of the 13-year-olds could measure the length of a segment to the nearest inch. However, when students were presented with a problem similar to the one illustrated in Figure II-2, 77 percent of the 9-year-olds and 40 percent of the 13-year-olds gave an answer of 5. Thus, although most students would line up the end of the segment when

**Figure II-2. How long is this line segment?**



they were measuring it, this change in problem context demonstrated that many of them did not understand the consequences of not doing so.

This superficial understanding was also apparent in many computation exercises. For example, students were relatively successful in multiplying two common fractions, perhaps because multiplying numerators and denominators seems to be a natural way to approach the problem. However, the results from the simple verbal problem shown in Figure II-3 indicate that students had no clear conception of the meaning of fraction multiplication and, therefore, could not apply their skills to solve even a simple problem.

**Figure II-3. Fraction Multiplication**

| Exercise* | Percent Correct | |
|---|---|---|
| | Age 13 | Age 17 |
| a) 2/3 × 2/5 = ? | 70 | 74 |
| b) Jane lives 2/3 mile from school. When she has walked 2/5 of the way, how far has she walked? | 20 | 21 |

* Both are similar to unreleased exercises.

Students' failure to learn basic fraction concepts is also illustrated by performance on several estimation exercises. For example, although 39 percent of the 13-year-olds and 54 percent of the 17-year-olds could calculate the answer to $\frac{7}{15} + \frac{4}{9}$, only 24 and 37 percent, respectively, could make even a reasonable estimate of $\frac{12}{13} + \frac{7}{8}$. Over half of the 13-year-olds and over a third of the 17-year-olds simply tried to add either the numerators or denominators. This response suggests that many students looked for some rote computation rule to apply without even considering the reasonableness of their result.

The importance of understanding may, in part, account for the difference in the level of performance for whole number operations and operations involving fractions and decimals. Most assessment exercises indicate that students had learned the basic concepts underlying whole number computation, and had some notion of the place value concepts involved in the computation algorithms. As a consequence, performance on whole number computation exercises was generally good. As the above examples suggest, however, most students did not have a clear understanding of fraction operations and appear to have operated at a mechanical level. This lack of understanding resulted in relatively poor performance on some fraction computation, and is further highlighted by the serious difficulties encountered in solving simple problems involving fraction operations.

The consequence of focusing on a mechanical application of basic skills is that students become totally dependent on a mechanical algorithm, which is easily forgotten. If students cannot remember a step in the algorithm, they cannot solve even simple problems that might be solved intuitively. For example, the complexity of problems (a) and (b) below appears to have had relatively little effect on their difficulty.

$$\text{(a)}\ \frac{1}{2}+\frac{1}{3} \qquad\qquad \text{(b)}\ \frac{7}{15}+\frac{4}{9}$$

In problem (a), students should have been able to find a common denominator almost intuitively. In problem (b), the denominators are not relatively prime and the least common denominator is 45. In spite of the difference in the apparent difficulty levels of the two problems, there was little difference in students' performance on them. Apparently, if students have learned an algorithm, they can apply it successfully in most situations. However, if they have not mastered an algorithm or have forgotten one step, they have difficulty with even simple problems that might be solved intuitively.

The results reported in this section suggest that many students have at best a superficial understanding of many mathematical concepts and processes. Yet around 90 percent of the 13- and 17-year-olds felt that developing understanding was an integral part of mathematics learning, as evidenced by their agreement with the statement "Knowing why an answer is correct is as important as getting the correct answer." Their responses may reflect their actual beliefs, or it may be that the statement was one they had heard from their mathematics teachers and was perceived as the expected response. The second alternative gains credence when considered in contrast to the fact that around 90 percent of both older age groups agreed that "There is always a rule to follow in solving mathematics problems." The students may be concentrating on mastering rules to the extent of ignoring concomitant understanding.

## Problem Solving

One of the consequences of learning mathematical skills rotely is that students often cannot apply the learned skills to solve problems. In general, NAEP results show that the majority of students at all age levels had difficulty with any nonroutine problem that required some analysis. It appeared that most students had not learned basic problem-solving skills, and attempted instead to mechanically apply some mathematical calculation to whatever numbers were given in a problem.

A marked discrepancy was found between the results of solution of routine and nonroutine problems. Students generally were successful in solving routine one-step verbal problems such as those often found in textbooks. The results summarized in Figure II-4 are representative of student performance on one-step verbal problems in which the main steps were deciding whether to add, subtract, multiply, or divide and then performing the calculation.

The verbal problem in Figure 4 was presented to 9- and 13-year-old respondents without a calculator and to another group of 9-year-olds who had a calculator available. A third set of respondents was presented the

**Figure II-4. Multiple-Choice Subtraction Exercises**

| Exercise* | Percent Responding | | |
|---|---|---|---|
| | Age 9 | | Age 13 |
| | Without calculator | With calculator | |
| a) George has 352 arithmetic problems to do for homework. If he has done 178 problems, how many problems does he have left to do? | | | |
| 174 (correct response) | 38 | 70 | 82 |
| 530 | 6 | 8 | 1 |
| 226 | 6 | 0 | 1 |
| Other subtraction error | 10 | 0 | 4 |
| b) 352<br>−178<br>? | | | |
| 174 (correct response) | 50 | — | 85 |
| 530 | 1 | — | 0 |
| 226 | 15 | — | 2 |

* Both are similar to unreleased exercises.

same subtraction calculation as a straight computation exercise. As with all exercises on the assessment that required reading, the verbal problems were presented on an audiotape as well as written in the exercise booklet. By age 13, there was very little difference in students' ability to solve the verbal problem and their ability to perform the required calculation. Furthermore, few students at either age level chose the wrong operation.

Although students could successfully solve most simple one-step problems, they had a great deal of difficulty solving nonroutine or multistep problems. In fact, given a problem that required several steps or contained extraneous information, students frequently attempted to apply a single operation to the numbers given in the problem. Students' difficulty with problems that could not be solved with a single operation is illustrated by the results summarized in Figure II-5. In spite of the fact that the problems involved calculations that were well within the students' range of computational skill, many were unable to solve either problem. In both problems, many students simply added or multiplied the numbers without analyzing the problem.

Students have not developed good problem-solving strategies. A basic strategy that helps in analyzing certain types of problems is to draw a pic-

**Figure II-5. Multi-Step Problems**

| Exercise | Percent Responding | | |
|---|---|---|---|
| | Age 9 | Age 13 | Age 17 |
| a) Lemonade costs 95¢ for one 56 ounce bottle. At the school fair, Bob sold cups holding 8 ounces for 20¢ each. How much money did the school make on each bottle?* | | | |
| Correct response | — | 11 | 29 |
| Students added, subtracted, or multiplied two of the numbers given in the problem. | — | 40 | 25 |
| b) Mr. Jones put a rectangular fence all the way around his rectangular garden. The garden is ten feet long and six feet wide. How many feet of fencing did he use? | | | |
| 32 feet (correct response) | 9 | 31 | — |
| 16 feet | 59 | 38 | — |
| 60 feet | 14 | 21 | — |

* Similar to an unreleased exercise.

ture. A problem related to the second exercise in Figure II-5 was given to another set of respondents. In this version, they were given a picture of the rectangle and asked to find the distance around it. This variation in the problem produced a difference of over 30 percentage points in the percent of correct responses for both 9- and 13-year-olds. Since most students could identify a rectangle as shown by results on a simple recognition exercise, many of them apparently did not apply their knowledge by drawing a figure to help them solve the verbal problem. Instead, many of them simply added or multiplied the two numbers given in the problem.

Another basic problem-solving strategy is to look for related problems that one knows how to solve to provide a method for solving the given problem. The results for two closely related problems suggest that students have difficulty transferring a solution method from one problem to even a closely related problem. Although most students could calculate the area of a rectangle, they were unable to recognize that a square simply represented a special case of the rectangle. About half of the 13-year-olds and three-fourths of the 17-year-olds could calculate the area of a rectangle, but only about 10 percent and 40 percent, respectively, could find the area of a square.

NCTM recommends that the development of the ability to solve problems be a major goal of school mathematics in the 1980s. The results of the second NAEP mathematics assessment suggest that we are a long way from achieving that goal. They also suggest a note of warning of how we should *not* approach that goal. Providing more experience with typical textbook verbal problems, while helpful, it not an adequate response to the recommendation. The assessment results indicate that in addition to teaching how to solve simple one-step verbal problems, more emphasis should be placed on nonroutine problems that require more than a simple application of a single arithmetic operation. Part of the cause of students' difficulty with nonroutine problems may result from the fact that their problem-solving experience in school has been limited to one-step problems that can be solved by simply adding, subtracting, multiplying, or dividing. The assessment results indicate that students have relatively little difficulty solving problems that only require them to choose the correct operation. In fact, their difficulties with nonroutine problems seem to result from their interpretation that problem solving simply involves choosing the appropriate arithmetic operation and applying it to the numbers given in the problem.

Instruction that reinforces this simplistic approach to problem solving may contribute to students' difficulty in solving unfamiliar problems. Although it may be argued that children must learn to solve simple one-step problems before they can have any hope of solving more complex problems, an overemphasis on one-step problems may only teach children how to routinely solve this type of problem. It may also teach them that they do not have to think about problems or analyze them in any detail.

Techniques designed to give children success with simple one-step problems that do not generalize to more complex problems may be counter-productive. For example, focusing on key words that are generally associated with a given operation provides a crutch upon which children may come to rely. Such an approach provides no foundation for developing skills for solving unfamiliar problems. Simple one-step problems may provide a basis for developing problem-solving skills, but only if they are approached as true problem-solving situations in which students are asked to think about the problem and develop a plan for solving it based upon the data given in the problem and the unknown they are asked to find.

Students need to learn how to analyze problem situations through instruction that encourages them to think about problems and helps them to develop good problem-solving strategies. Students need ample opportunity to engage in problem-solving activity. If problem solving is regarded as secondary to learning certain basic computational skills, many students are going to be poor problem solvers. Additional discussion of implications of the NAEP results for problem solving at the elementary and secondary levels can be found in Carpenter and others (1980a, 1980d).

## Continued Development of Mathematical Skills

Although problem solving and many noncomputational skills clearly require an increased emphasis in the curriculum, we do not deny the importance of computational skills. A reasonable level of computational skill is required for problem solving. We are suggesting, however, that problem solving not be deferred until computational skills are mastered. Problem solving and the learning of more advanced skills reinforce the learning of computational skills and provide meaning for their application.

It is important to recognize that most computational skills are learned over an extended period of time. The results summarized in Figure II-6 suggest that most skills are mastered after their period of primary emphasis in the curriculum. For example, even though a goal of most mathematics problems is that students learn to subtract by age 9, there was significant improvement in performance on subtraction exercises from age 9 to 13 and there was even some improvement between ages 13 and 17. Furthermore, many fundamental errors also disappear as students progress in school. Although over 30 percent of the 9-year-olds subtracted the smaller digit from the larger in a subtraction exercise that required regrouping, only 5 percent of the 13-year-olds and 1 percent of the 17-year-olds committed this error.

**Figure II-6. Improvement in Performance by Age**

| Problem | Percent Correct | | |
|---|---|---|---|
| | Age 9 | Age 13 | Age 17 |
| a) Basic subtraction facts | 79 | 93 | 95 |
| b) Three-digit subtraction | 50 | 85 | 92 |
| c) $4/12 + 3/12 = ?$ | — | 74 | 90 |
| d) $1/2 + 1/3 = ?$ | — | 33 | 66 |

These results have profound implications for minimum competency programs. Rigid minimum competency programs which hold children back until they have demonstrated mastery of a given set of skills may, in fact, be depriving them of the very experiences that would lead to mastery of the particular skills.

Although some skills will continue to develop through use in other contexts, this is not always the case. The current high school curriculum does not take into account that many basic skills are not well-developed by the time students begin instruction in algebra and geometry. For example, very few 13- and 17-year-olds have mastered percent concepts or skills, but outside of general mathematics classes, there is very little opportunity for high school students to extend or maintain their knowledge of percent.

## Implications of Calculators for Teaching Computational Skills

Over 85 percent of the 17-year-olds in the assessment indicated that they had access to a calculator. This availability of calculators would seem to have profound implications for the appropriate level of emphasis that computation should receive and the types of algorithms we should teach.

In spite of the extensive instruction provided on whole number division, only half of the students assessed were reasonably proficient in division by the time they were ready to graduate from high school. With a calculator, however, over 50 percent of the 9-year-olds and over 90 percent of the 17-year-olds could divide accurately. This raises some serious questions as to whether the time spent drilling on division is a productive use of time and effort that might otherwise be devoted to other topics. Certainly, it is clear that our current approach to teaching division is not effective for a substantial number of students.

The division algorithm as well as most of the other algorithms that we teach in school are designed to produce rapid, accurate calculation procedures. Given the widespread availability of hand calculators, it would seem that the continued emphasis on developing facility with computation algorithms should not be as high a priority as it was formerly. Certainly, computation is important; but what is needed are algorithms that students will remember and will be able to generalize to new situations. This brings us back to the issue of understanding. Students are more likely to remember and be able to generalize and apply algorithms if they understand how the algorithms work. Thus, it may be appropriate to begin to shift to computational algorithms that can be more easily understood than the ones currently taught, even if they are less efficient.

The results for the following problem illustrate the potential impact of calculators on our thinking about computation:

> A man has 1,310 baseballs to pack in boxes which hold 24 baseballs each. How many baseballs will be left over after the man has filled as many boxes as he can?

Students had more difficulty solving this problem with a calculator than without using a calculator. Twenty-nine percent of the 13-year-olds correctly solved this problem without a calculator while only 6 percent of the 13-year-olds who had a calculator were successful. Students also had more difficulty comparing and ordering a set of fractions with a calculator than they did without one. They apparently did not understand that fractions can also be thought of as quotients, which allows one to represent them as decimals that are relatively easy to order. These results suggest that students have rigid ways of thinking about numbers and operations. Calculators sometimes require alternative interpretations and require that students have a deeper understanding of numbers and how the operations work.

Calculators also place an increased importance on estimation skills and alertness to reasonableness of results. The results of one exercise given to 13- and 17-year-olds illustrate the importance of this skill and the gross errors that can occur when students using a calculator are oblivious to the reasonableness of a result. Students were asked to divide 7 by 13 using a calculator. About 20 percent of the 13- and 17-year-olds chose the response 5384615 rather than 0.5384615.

## Participation in Mathematics Courses

Mathematics learning is a continuous process that encompasses the entire 12 years of elementary and secondary school. Many, if not most, basic skills are not mastered by age 13 and must be reinforced and developed as part of the high school curriculum. Consequently, if we are going to significantly improve the mathematics performance of high school graduates, we must ensure that they continue to take mathematics throughout their high school program. The NCTM (1980) has proposed that at least three years of mathematics should be required of all students in grades 9 through 12. The assessment background data summarized in Figure II-7 indicate that we are currently far short of that goal.

**Figure II-7. Mathematics Courses Taken by 17-Year-Olds**

| Course | Percent having completed at least ½ year |
|---|---|
| General or Business Mathematics | 46 |
| Pre-Algebra | 46 |
| Algebra I | 72 |
| Geometry | 51 |
| Algebra II | 37 |
| Trigonometry | 13 |
| Pre-Calculus/Calculus | 4 |
| Computer Programming | 5 |

## Student Perceptions of Mathematics Classes

Among the recommendations of NCTM for the curriculum of the 1980s are several statements that indicate a need for teachers to encourage experimentation and exploration by students as part of the requisite atmosphere that encourages problem solving. Included is a call for teachers to "provide ample opportunities for students to learn communication skills in mathematics" (NCTM, 1980, p. 8) in both reading and talking about mathematics, and a recommendation to teachers to incorporate "diverse instructional strategies, materials, and resources, such as—individual or small group work as well as large group work; . . . the use of manipulatives (where appropriate); . . . the use of materials and references outside the classroom" (pp. 12-13). The implication from these recommendations is that mathematics teachers should provide opportunities for their students to be actively involved in learning and communicating mathematics.

One set of exercises attempted to assess how often students engage in different activities in mathematics classes. Students were presented with a list of classroom activities and asked to rate them in terms of how often they thought the activities occurred in their mathematics class. The activities may be described as student-centered, teacher-centered, classmate-centered, and "other," which included activities requiring active student involvement such as using manipulative objects.

**Figure II-8. Ratings of Frequency of Selected Classroom Activities**

| Activity | Percent Responding | | | |
|---|---|---|---|---|
| | Age | Often | Sometimes | Never |
| *Student-centered* | | | | |
| Mathematics tests | 9 | 44 | 46 | 9 |
| | 13 | 61 | 37 | 1 |
| | 17 | 63 | 33 | 3 |
| Mathematics homework | 9 | 43 | 45 | 12 |
| | 13 | 67 | 29 | 3 |
| | 17 | 57 | 36 | 6 |
| Worked mathematics problems alone | 9 | 71 | 22 | 7 |
| | 13 | 81 | 17 | 9 |
| | 17 | 80 | 19 | 1 |
| Worked mathematics problems on the board | 9 | 39 | 54 | 7 |
| | 13 | 33 | 58 | 9 |
| | 17 | 27 | 60 | 13 |
| Used a mathematics textbook | 9 | 75 | 18 | 6 |
| | 13 | 81 | 14 | 5 |
| | 17 | 87 | 11 | 3 |
| Worksheets | 9 | 71 | 27 | 2 |
| *Teacher-centered* | | | | |
| Listened to the teacher explain a mathematics lesson | 9 | 85 | 11 | 3 |
| | 13 | 81 | 16 | 2 |
| | 17 | 78 | 19 | 2 |
| Watched the teacher work mathematics problems on the board | 9 | 78 | 19 | 4 |
| | 13 | 76 | 21 | 2 |
| | 17 | 79 | 18 | 3 |
| Received individual help from the teacher on mathematics | 9 | 21 | 67 | 11 |
| | 13 | 17 | 71 | 10 |
| | 17 | 18 | 70 | 11 |

Figure II-8 lists the frequency ratings on the student- and teacher-centered activities. Table headings represent response options for 13- and 17-year-olds to the question "How often have you done these activities in your mathematics classes?"; options for the 9-year-olds were "a lot," "a little," and "never." As the table shows, most students reported that they spent a lot of time listening to and watching the teacher work and explain mathematics problems. They also reported that they spent a lot of time working alone on mathematics problems from the textbook and, for the 9-year-olds, also from worksheets. The percentages of frequency assigned to these activities were the highest for any of the group of activities.

As a group, the classmate-centered and "other" activities received the highest percentages of "never" ratings for all age groups. Among the classmate-centered activities, around half of all age groups said that discussing mathematics in class occurred often; around 60 percent of the 9-year olds and 75 percent of the 13- and 17-year-olds said they sometimes gave help to or received help from their classmates in mathematics. Thirty-five, 44, and 28 percent of the 9-, 13-, and 17-year-olds, respectively, said they never worked mathematics problems with small groups of students.

Most of the older students said they had never made reports or done projects in mathematics classes, and over two-thirds of the 9-year-olds and three-fourths of the older respondents said they had never done mathematics laboratory activities. Further, over half of the 9-year-olds said they had never used objects like counters, rods, or scales in mathematics classes.

These National Assessment results show that students perceive their role in the mathematics classroom to be primarily passive. They are to sit and listen and watch the teacher do the problems; the rest of the time is to be spent working on an individual basis on problems from the text or from worksheets. They feel they have little opportunity to interact with their classmates about the mathematics being studied, to work on exploratory activities, or to work with manipulatives.

An attempt to evaluate the implications of these results for the curriculum of the 1980s leads directly to the issue of the extent of student involvement in the learning process. The results suggest that the current situation, at least from the students' point of view, is one in which mathematics instruction is "show and tell" on the teacher's part, "listen and do" for the students. Students' perception of their involvement is in direct contrast to the recommendations of NCTM. If active student involvement in mathematics learning is as desirable and sought after as the NCTM recommendations imply, then changes in approaches to teaching mathematics that will foster and encourage that involvement must be implemented.

## Closing Thoughts

Undoubtedly, it will take at least the entire decade to make significant progress in fully implementing NCTM's recommendations for the 80s. Thus, these recommendations represent goals to strive to achieve by the end of the decade.

The National Assessment results provide one measure of where we are at the beginning of the decade. They also suggest that the development of routine computational skills has been the dominant focus of the school mathematics curriculum, and that the development of problem-solving skills has been inadequate.

Although we are a long way from the kind of program envisioned in the NCTM recommendations, the assessment results provide some basis for cautious optimism. It is probably fair to say that the focus of mathematics instruction has been on computation. There is evidence that students are learning what they are being taught. There is also evidence that curricular reforms can have some impact. On exercises that measured change in performance from the first assessment, there were significant gains of 10 to 20 percentage points on exercises that dealt with metric measurement. These results appear to reflect the increased emphasis on metric measurement in the curriculum over that period of time.

Improved student performance in mathematics is a goal that demands the combined efforts of many people. The results presented here have shown that there is room for much improvement, but there is hope that if we can reorganize the mathematics curriculum to address the NCTM recommendations, students' performance will respond accordingly.

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# Response

Diana Wearne

The National Assessment of Educational Progress has provided a wealth of information about the mathematics achievement of elementary and secondary students. Carpenter and his colleagues derived a number of significant conclusions from the NAEP data about students' current level of achievement and have offered several noteworthy suggestions for improving school mathematics programs.

Interpreting the results of any test is difficult, particularly when the test is of the magnitude of National Assessment. Several cautions must be exercised in reading reports, some of which are: (1) selections of specific items to interpret may affect the conclusions; (2)

items may not assess the stated objectives; and (3) implications of results may depend on value judgments about the educational importance of individual items.

Whenever sets of items are discussed and it is not possible for the reader to see all of the items (the NAEP tests had between 250 and 430 items at any age level), there exists the possibility that analyzing different sets of items may lead to different conclusions. The authors of the chapter were careful to caution readers as to this possibility.

It is difficult to err in constructing items that assess computation; however, assessing less specific and definable objectives can be difficult. In an effort to assess teaching methods, students were asked to indicate how often they had participated in specific categories of activities, but not the nature or characteristics of the activities. Some 43 percent of the 9-year-olds indicated they often had homework. The homework could have consisted of a page of computation or it could have involved searching for information to record, organize, and use the following day in graphing activities. These are distinctly different types of homework and it is impossible to determine what percentage was of each type.

It also is possible the students' perceptions of how much time was devoted to a given topic or activity were confounded by their attitudes. A 13-year-old who is assigned homework once a week but dislikes it may feel he or she is always having to do homework. A student who has not been in a mathematics course for a year or more may remember only the activities that were especially pleasant or distasteful.

Two-thirds of the 9-year-olds reported they had never participated in mathematics laboratory activities. Some students may have been involved in such activities but did not realize it. For instance, children involved in measurement activities designed to yield numbers for computing may not have considered that a laboratory activity. Half of the 9-year-olds indicated they had never used counters, rods, or scales. They may have used counting manipulatives but called them by other names, such as "popsicle sticks." A child who was involved in an activity for only a portion of the class time and who also completed a workbook page may have responded that textbooks were used always or often.

Another possible misinterpretation of the test results relates to the time at which the tests were administered. Carpenter and others cite the data in Table 20 as evidence that "we are currently far short" of the goal of three years of mathematics for all students in grades 9-12. When the tests were administered in March or April, 72 percent of the 17-year-olds were in the eleventh grade. Some of those students undoubtedly enrolled in a mathematics course in their senior year, increasing the actual percentage of students who took more advanced courses during their high school career.

On another topic, the authors caution against giving too much emphasis to one-step application problems. That may be good advice from a problem-solving point of view, but one-step problems give meaning to mathematical operations where it may otherwise be lacking, as apparently is the case with fractions. Young children understand addition and subtraction within a verbal, one-step problem context before they develop meaning for the symbolic representation.[1] Perhaps a similar link between verbal problem situations and their symbolic representations would help children develop meaning for other operations (multiplication and division) and other kinds of numbers fractions, negative numbers). The one-step problems should force children to think about the meaning of the operations in the verbal context and should not be solved as a series of routine calculations.

Carpenter and others have painted both a pessimistic and an optimistic picture of mathematics learning in the United States. It is pessimistic because it shows what students are not learning but optimistic because it shows that positive changes have taken place. Citing improved performance in metric measurement because of added emphasis to this topic since the previous assessment, the authors say that pinpointing deficiencies may result in further improvement in student performance. I am not as optimistic.

The pressure to include metric measurement in textbooks came both from inside and outside the mathematics community and change was relatively easy to affect. Improving problem-solving skills is much more complex. Redesigning textbooks so they develop understanding and involve students in problem-solving activities is more than a cosmetic change of including a few additional pages in a book. And needed changes on the part of teachers and administrators are even more difficult to achieve.

One reason for cautious optimism may be the number of well-attended sessions on problem solving at regional and national meetings. Perhaps teachers' interest and continuing emphasis on this important topic by organizations will bring improved problem-solving performance on the next National Assessment.

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[1] Carpenter, T. P. "The Effect of Instruction on First-Grade Children's Initial Solution Processes for Basic Addition and Subtraction Problems." Paper presented at the annual meeting of the American Educational Research Association, Boston, April 1980.

# III. Children's Thinking

James Hiebert

Educators and psychologists have long been interested in children's ability to deal with mathematical tasks. Since the traditional research approach has focused on the outcomes of children's performance, investigations are often designed to determine the effects of certain external conditions on these outcomes. For example, a typical research study might investigate the effects of two different methods of instruction on whole number addition by comparing the number of correct responses on an addition test after instruction. The answers children give are used to infer something about the effectiveness of the external conditions under which the concepts were learned.

In contrast to the concern with conditions outside of the learner and the focus on performance outcomes, several lines of current research are looking directly at the *processes* children use to solve mathematical problems. Consistent with current trends in cognitive psychology, this research focuses on the things that occur inside the child's head. Of course, we cannot actually see inside the mind, so many of the conclusions are based on inference. But the strategies or processes children use provide a window on their thinking. In some areas, where research efforts have been quite intensive in recent years, it is possible to paint a reasonably good picture of children's thinking.

There are two levels at which this type of research on children's mathematical thinking has been carried out. One is an underlying, funda-

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mental level concerned with general principles of thinking and learning. At this level, research focuses on basic cognitive processes that potentially are involved in dealing with a wide range of mathematical situations. Piaget's work is of this kind. The second level deals with processes that are specific to similar types of mathematical problems. Of primary interest are the strategies that children use to solve a given class of problems. Detailed descriptions are given of what children do when they solve linear measurement problems, when they add two whole numbers, when they reduce fractions, or when they solve any other specific type of mathematical problem.

In this chapter we will review and synthesize some of the research at each of these two levels, and look at how the research findings might be used in the design of better instructional programs. Piaget's work and some recent developments in information processing theory provide the basis for the review of general cognitive processes. Several areas of research could be used as examples of the work on more specific mathematical processes. Research on early number concepts and initial arithmetic operations was selected for this review. Children's thinking in this area has received much attention in recent years and the work nicely illustrates this research perspective. It is important to note that this chapter will only sample from the existing research on children's thinking. Readers may also wish to consult other reviews of children's mathematical thinking that are either more extensive, focus on other content, or have been written from a different perspective (for example, Brainerd, 1979; Carpenter, 1976, 1979; Lesh and Mierkiewicz, 1978; Shumway, 1980).

## General Cognitive Processes

An abundance of research has been carried out within the past decade in an attempt to uncover relationships between various cognitive processes and performance on mathematical tasks. Two types of cognitive processes have appeared to be most closely related to mathematics learning and have dominated the research in this area. These are the logical reasoning abilities described by Piaget, and information processing capacity, as characterized by recent work in cognitive psychology (Campione and Brown, 1979; Case, 1978a, 1978b).

### Logical Reasoning Abilities

In studying how children acquire knowledge and learn about the world, Piaget and associates (Piaget, 1952; Piaget, Inhelder, and Szeminska, 1960) looked very closely at children's thinking. They found many things that surprised them. If they laid out one row of candies and the child laid out a second row with the same number of candies, and then they spread the candies in one of the rows, the child would respond that now the longer row had more candies. Through continued observations of children, Piaget discovered that this response was typical of all young children. Before a certain age (usually around five or six years) children do not conserve number, that is, they do not recognize that moving the objects in a set has no effect on the number of objects the set contains. After finding similar nonconservation responses on tasks with other quantities, such as length and area, Piaget concluded that conservation is a hallmark in the development of logical reasoning. It represents a fundamental difference between children's thinking and adults' thinking.

Educators and psychologists have pointed out that the logical reasoning abilities identified by Piaget, such as conservation, may be essential for solving a variety of mathematical problems (Elkind, 1976; Lesh, 1973). An analysis of many mathematical tasks shows that these abilities seem to be logical prerequisities. For example, solving a simple addition or subtraction problem using concrete objects involves moving the objects about and regrouping them in various ways. Many of the strategies involve transformations on objects that logically presuppose the ability to conserve number. A similar analysis, applied to a variety of measurement tasks, suggests that conservation of length may be required to learn foundational concepts of measurement.

While the prerequisite relationships between conservation and learning related mathematical concepts seem quite logical from an adult perspective, they have been difficult to document with children. Recent history has recorded a continuing debate among researchers about the importance of conservation for learning mathematics. Many of the early studies found a general relationship between passing conservation tasks and scoring well on mathematics achievement tests, but some conflicting evidence was also reported. In many ways this early research raised as many questions as it answered (Carpenter, 1980). Recently, investigators have become more sophisticated in their approach to this problem and have resolved many of the previous questions. The success of these studies is due in part to the fact that most of them have focused on specific mathematical concepts rather than on general achievement; some have followed children's progress over a carefully designed instructional sequence; and many have considered the processes children use to solve problems rather than looking only at correct and incorrect responses. These methodological improvements have helped to illuminate the role of conservation in children's ability to learn mathematics.

Arithmetic and measurement have provided the arena for much of the recent research on the relationship between conservation and mathematics learning. Several initial studies reported a significant relationship between number conservation and performance on certain kinds of addition and subtraction problems (LeBlanc, 1971; Sohns, 1974; Steffe, 1970; Steffe and Johnson, 1971). But no clear pattern emerged that might suggest which arithmetic problems depend on conservation and which do not. More recent studies have provided at least a partial answer to this question. Mpiangu and Gentile (1975) taught kindergarten children a variety of simple counting and number skills, and found that nonconservers gained as much from the instruction as conservers. Although the conservers performed more successfully than nonconservers, both before and after instruction, their equal gains led to the conclusion that conservation is not needed to learn arithmetic skills.

A later study by Steffe and others (1976) confirmed that nonconservers can learn simple skills but their results suggest that conservation may be important for solving more complex arithmetic problems. After several months of instruction on various counting strategies, nonconservers had trouble applying them to solve missing addend problems, while conservers were quite successful. Steffe and others argue that conservation is not needed to complete addition and subtraction problems that can be solved with simple counting skills, but it is important for understanding the more complex problems, like those with missing addends.

A study with first-grade children by Hiebert and others (1980) provides further evidence that nonconservers can solve a variety of verbal arithmetic problems. While there were significant differences in the accuracy with which conservers and nonconservers solved some of the problems, there were, for each problem type (addition, subtraction, and missing addend), a number of nonconservers who responded correctly. Furthermore, each kind of solution strategy was used by at least some nonconserving children. In fact, the frequency with which nonconservers applied the more advanced strategies did not differ significantly from their conserving peers. The picture that emerges from these results, along with those of previous studies, is that conservation is not needed to learn elementary counting skills nor to solve simple verbal or symbolic addition and subtraction problems. While it may facilitate performance on missing addend problems there is good reason to believe that it is not a prerequisite.

Research on children's learning of measurement yields results that are similar to those for arithmetic. Several studies have investigated the sequence in which a variety of measurement concepts are acquired. If the ability to conserve is needed to learn certain concepts or skills, then one would expect successful performance on a conservation task to precede mastery of the concept or skill tasks. Apparently this sequence does occur for measurement tasks that assess children's understanding of the inverse relationship between unit size and unit number, that is, the fact that more units are needed to measure a given quantity if they are small than if they are large (Bradbard, 1978; Carpenter, 1975; Hatano and Ito, 1965; Wohlwill, 1970). However, there are many measurement skills that precede the appearance of conservation. This group included: (1) the ability to iterate units, e.g., move a single unit across a surface to measure its length (Bradbard, 1978); (2) proficiency in applying standard measurement techniques, such as using a ruler to measure length (Hatano and Ito, 1965); and (3) the ability to attend to the number of units measured and infer that the quantity which measured the most units is the largest (Carpenter, 1975; Wagman, 1975).

The results of these studies show that nonconservers learn a variety of measurement skills, but they do not indicate what the limits of this learning might be. To obtain this information, Hiebert (1981) instructed length conserving and nonconserving first-grade children on several basic concepts of linear measurement. At least some nonconservers learned each of the concepts and skills except one—using the inverse relationship between unit number and unit size to construct a length. On all tasks but this one, nonconservers used the same kind of solution strategies as their developmentally advanced peers. Apparently conservation is a true prerequisite for this one concept of measurement, but is not needed to master many other measurement concepts.

The argument for using conservation as a readiness measure for instruction is based on the assumption that conservation is a prerequisite for learning various mathematical concepts or skills. Since conservation is not easily taught, and since it presumably represents a fundamental logical reasoning ability, it may be better to postpone instruction on these concepts until the reasoning ability develops. That's the logical argument. Its validity obviously rests with being able to establish empirically that certain mathematical tasks do, in fact, require conservation to solve them.

The research reviewed here focused on the role of conservation in learning initial arithmetic and measurement concepts. The evidence suggests that conservation tasks are of limited value as readiness measures for instruction on these concepts. Except for the concept of the inverse relationship between unit number and unit size in measurement situations, conservation does not seem to be essential for learning to solve school mathematics tasks. There are simply too many children who fail the conservation tasks and perform successfully on the mathematics tasks.

The problem is that even though conservation is a *logical* prerequisite for completing many arithmetic and measurement tasks, children do not seem to use conservation knowledge when they solve the tasks. Children's solution procedures are different than the structural logic of the problem. They move and regroup objects to solve a simple addition problem and do not think to ask the conservation question; they simply count the objects to find the answer. They move a unit to measure the length of an object and do not worry about whether the length of the unit is being conserved. Children seem to focus only on the question at hand and do not recognize that conservation provides an essential logical foundation for the task. Simple skills, such as counting, apparently allow children to bypass the logical structure of many mathematical tasks.

Conservation is not the only reasoning process described by Piaget that potentially affects mathematics learning. Transitive inference and various classification skills are also believed to be fundamental thought processes that support the acquisition of many mathematical concepts (Piaget, 1952; Piaget and others, 1960). However, here too the available research evidence suggests that these abilities are not prerequisites for dealing successfully with logically related mathematics tasks (Hiebert, 1981; Hiebert and others, 1980; Sohns, 1974; Steffe and others, 1976). Researchers may be more successful in establishing relationships between the more advanced, formal reasoning processes in Piaget's theory and the mathematics learning of adolescents (Adi, 1978; Carpenter, 1980). Many of the school mathematics tasks at this level seem to involve directly the abstract reasoning skills measured by the formal reasoning Piagetian tasks. However, so little research has been done at this level that it would be inappropriate to speculate on the nature of these relationships.

## Information Processing

The basic notions of information processing theory grew out of a concerted attempt to describe what the learner actually does when solving a problem or acquiring a new skill. The objective is to describe how the learner processes information, and then to use these descriptions to build models of the human information processing system. Many times these models are precise enough so that they can be written in computer language. In this case the validity of the model can be checked by giving the computer and a student the same problem and observing how closely the computer simulates the performance of the student. The value of building these models is that a great deal of thought must be given to detail the processes that are used to solve a particular problem. Computers don't work well unless they are programmed with precision. Consequently, the models help to identify some of the critical points in the thinking process.

Most information processing models include three important features. One is a component labeled working or short-term memory. This is the center of all thinking or information processing. It has a limited capacity, which is usually described in terms of the number of separate pieces of information that can be processed at the same time. A second feature of most models is a description of the processes that could be used to solve a given class of problems. If computer simulations are developed, these processes are described in great detail. A third feature of recent models is a planning and organizing function which serves to oversee the actual processing of information. The so-called meta-cognitive processes include plans and strategies to decide what pieces of information to focus on at any one time, how to organize the information so that it can be processed most efficiently, and which of all available strategies can best solve the problem. While the first feature of the model is concerned primarily with the limits of the system, the second and third have focused more on its capabilities.

**Capacity Limitations on Learning.** Short-term memory is a critical part of the information processing system because there is a definite limit on the number of information bits that can be handled simultaneously. Try adding 275 and 468 in your head without referring back to the numbers. Even though you know all the rules for completing this simple problem it probably puts some strain on your information processing system. If you made a mistake in computing the answer you can probably put the blame on insufficient short-term memory capacity. Children experience even greater difficulty with these kinds of problems because they have a more restricted capacity. Young children can process only about one-fourth to one-half the number of information pieces that adults can handle.

Some researchers have suggested that children's restricted processing capacity has considerable consequences for the curriculum because it may place severe constraints on children's ability to profit from instruction. Instructional tasks require children to receive, encode, and integrate information. In many cases, children may possess all of the necessary skills for a particular task and still fail the task. The reason for this failure may be children's restricted capacity to deal with all of the information needed to complete the task (Case, 1975).

The research to date has shown that information processing capacity does constrain children's learning to a predictable degree on specially-designed laboratory tasks (Case, 1974). However, it has been more difficult to isolate the effects of this capacity on school mathematics tasks (Hiebert, 1981; Hiebert and others, 1980). Recent work in this area suggests that part of the problem in identifying capacity constraints lies in developing valid and reliable measures of processing capacity (Romberg and Collis, 1980a, 1980b). Some progress is being made in constructing measures but their application to instructional settings is still far from being realized.

**Processes and Meta-Processes.** Although it is reasonable to think that certain underlying cognitive capacities are needed to learn mathematics, and that an insufficient capacity would limit children's learning, the research on what children *cannot* learn has not been very productive. Recently, a number of investigators have begun looking at the general cognitive processes children do have, and the ones they are capable of learning. These include both the processes that are used to solve problems, and the meta-processes that serve to select and monitor the execution of specific procedures. While short-term memory capacity develops with maturity and cannot be readily improved by specific training, the processes of the system are influenced by instruction. Individuals can be taught strategies that process information more efficiently and push back the limits that might otherwise be imposed by their restricted processing capacity (Brown, 1978; Brown and others, 1981).

It is too early to tell what implications the research in this area might have for the mathematics curriculum. However, several characteristics of this approach are already being used with impressive success to study children's thinking in mathematical situations. One important characteristic of this approach is the emphasis on the proficiencies children have. Children are viewed as capable thinkers who have at least the rudiments of effective problem solving. The assumption is that the things children do make sense to them, and attempts are made to describe in detail, from the children's perspective, the processes they use to deal with information.

A second important characteristic of the recent information processing approach is the concern with careful task analyses. Children may perform quite differently on several tasks measuring the same concept because of differences in task format, the type of response required, or other task variables. Children's real competencies may be hidden by irrelevant task variables. Understanding the task is essential for understanding the processes children use to solve the task. Therefore, a variety of task analysis procedures are used to describe the underlying structure of tasks as well as surface characteristics that may affect performance.

A third significant characteristic of recent research in information processing is the focus on local, rather than global, processes. It has been very difficult to find general principles of learning and thinking that apply to a wide variety of situations. Recognizing this problem, researchers are turning their attention to specific processes that are used on a well-defined set of similar tasks. They believe that it is more productive at this point to describe in detail the particular strategies that are used to solve a homogeneous class of problems than to continue searching for general processes that are involved in a wide range of problems.

## Children's Thinking About Number and Arithmetic Concepts

One line of research within the mathematics education community that has successfully applied the general research perspective arising from the information processing approach is the study of number and arithmetic concepts. Recent work in this area has been directed toward describing what young children know about number and arithmetic operations, even before they receive formal instruction; how task variables affect their performance; and what strategies they use to solve different types of arithmetic problems. Although the picture is not yet complete, we are able to describe some important pieces of children's thinking in this area. Consequently, this line of research was selected as an example of current research on children's mathematical thinking. For discussions of research in other areas of mathematics learning and thinking see recent reviews edited by Shumway (1980), Lesh and Mierkiewicz (1978), and Lesh and others (1979).

### Development of Early Number Concepts

Children achieve an initial concept of number through counting. Although this conclusion may seem obvious to those who have observed young children answer questions of *how many?*, a series of recent studies has shown how important, and how complex, the counting process is. Fuson (1979, 1980; Fuson and Mierkiewicz, 1980) and Steffe (Steffe and others, 1976; Steffe and Thompson, 1979) trace the development of children's ability to count from when they first verbalize a string of number words to the point where they can use efficient counting techniques to solve a variety of arithmetic problems. An analysis of the counting act shows that a process as simple as finding "how many" objects there are in a set involves the coordination of several separate actions: saying the number word string beginning with one, and pointing to a different object as each number word is spoken.

As children's counting proficiencies continue to develop, two major breakthroughs can be identified. The first occurs when children begin to establish relations among the counting words rather than producing the number word string as a single unit. A symptom of this new facility is that children can now count forward from a number other than one, or count back from a given number. They can give the number that comes just before, or just after, a number without counting from one. A second significant advance occurs when children recognize that the number words themselves can be used as the objects of counting. This new realization substantially increases the power of the counting process in solving problems. For example, children can now count on to find the number that is 5 more than 8 by counting the number words after 8 and stopping at the fifth one. This represents a change in *what* is counted. Concrete objects are no longer needed as counters; number words can serve as the unit items.

The act of counting rests upon several important principles. Described in detail by Gelman and Gallistel (1978), they include the fact that each object to be counted must be assigned one and only one number word, that the same number list must be used every time a set of objects is counted, that the last number word gives the numerosity of the set, and that the order in which the objects are counted does not matter. Gelman (1977, 1978; Gelman and Gallistel, 1978) argues that counting is a natural process for young children, and that before entering school they already understand these principles. According to Gelman, learning how to count is primarily a matter of learning the standard number words (one, two, three . . .) and applying the principles to larger and larger numbers.

### Development of Addition and Subtraction Concepts

Along with their counting skills, many preschool children develop some sound, intuitive ideas about arithmetic operations. For example, by four years of age, most children understand that addition increases numerosity and subtraction decreases numerosity, even though they may have trouble calculating the numerical outcome of the increase or decrease (Brush, 1978). If the sets are small enough so young children can count them, many children also seem to recognize that addition and subtraction are inverse operations in the sense that the effect of one cancels the effect of the other, and that, if trying to keep two sets equivalent, adding objects to one set can be compensated for by adding objects to the other set (Gelman and Gallistel, 1978; Gelman and Starkey, 1979). These intuitive notions, together with effective counting skills, provide children with a significant fund of knowledge with which to begin school.

Children's interpretations of arithmetic progress as they receive instruction, but in the first few years this progress appears to be closely tied to the development of their counting abilities. Ginsburg (1977b) believes that counting is so important for children that even after formal instruction "the great majority of young children interpret arithmetic as counting" (p. 13). It is certainly true that before children learn basic addition and subtraction facts they solve arithmetic problems by counting. A series of studies at the Wisconsin Research and Development Center for Individualized Schooling (Carpenter and others, 1981; Carpenter and Moser,

1979) and at the Pittsburgh Learning Research and Development Center (Heller and Greeno, 1979; Riley and Greeno, 1978) have shown that first-grade children, even before receiving instruction, can solve verbal addition and subtraction problems by applying appropriate counting strategies. Furthermore, they do not use the same strategy to solve every problem, but have available a rich repertoire of strategies and use different strategies to solve different types of problems.

One objective of this research has been to determine what factors affect the kinds of strategies that children use to solve different types of verbal addition and subtraction problems. The one factor that consistently stands out as the most significant in this regard is the type of action or relationship between the sets described in the problem (Carpenter and others, 1981a; Riley and Greeno, 1978). The importance of this "semantic structure" is best explained by considering several sample problems. The following problems are all solvable by subtracting the smaller number from the larger: (1) John has 8 apples. He gave 5 apples to Mary. How many apples does John have left? (2) John has 5 apples. Mary gave him some more apples and now he has 8 apples. How many apples did Mary give to John? (3) John has 8 apples. Mary has 5 apples. How many more apples does John have than Mary? If first-grade children are provided with physical objects to be used as counters, and are read these three stories, the majority of children will solve each problem using a different counting strategy. Almost all children who use the counters will solve the first problem by making a set of eight, removing five, and counting the rest. Most children will solve the second problem by making a set of five, adding on additional markers by counting "six, seven, eight," and then counting the number of markers added on. While there is more variation on the third problem, many children will count out a set of five and a set of eight, match the two sets using a one-to-one correspondence, and then count the unmatched markers in the larger set.

It is clear from these examples that many children solve the problems directly by carrying out the action or representing the situation that is described. Although at first glance this may not appear to be a particularly profound conclusion, it carries with it at least two potentially important implications. First, it means that even before receiving instruction, children are sensitive to the critical verbal cues in a story that indicate what action is appropriate to solve the problem. At this point in the learning process, very few children apply the wrong operation to solve a problem. That is, in general they do not add when they should subtract, or subtract when they should add. Not only do they carry out the correct operation, they often match their strategy to the context or semantic structure of the problem.

The fact that initially children use different strategies to solve different types of subtraction problems leads to the second important point. Apparently children perceive these different subtraction situations to be genuinely different problems. Although adults can see the commonality in these problems, and recognize that they can all be solved using the same "subtraction" procedure, it seems that many young children do not possess such a general subtraction concept. They see these as different problems that are solvable by different methods.

## Development of an Arithmetic Symbol System

As children proceed in school, they receive instruction in the formal symbolism of arithmetic. They are taught to represent the verbal problems presented earlier as $8 - 5 = \square$. It is at this point that many children experience difficulty. Writing the same number sentence for these three different problems requires children to see them as mathematically similar, an expectation which may go beyond the knowledge and capabilities of first- and second-graders (Gibb, 1956; Vergnaud, 1979).

In addition to the problems children may have with collapsing their many different interpretations of arithmetic situations into a single addition category and a single subtraction category, they also seem to experience difficulty in relating the verbal problem to a symbolic equation, of whatever kind. At the end of first grade, many children can solve verbal addition and subtraction problems, and some can write number sentences which represent these problems. But Carpenter and others (1981b) found that many children view these two processes as being independent. Children in this study often wrote the symbolic equation after, rather than before, finding the solution. The act of writing a number sentence rarely influenced the choice of a solution strategy.

At the heart of this problem is the fact that young children are not always able to give meaning to the formal, arbitrary symbolism of mathematics. Lindvall and Ibarra (1980) report that first- and second-graders have a difficult time demonstrating with concrete objects the meaning of a simple addition or subtraction equation. Grouws (1972) found that even in the third grade, many children did not solve addition and subtraction number sentences when the position of the unknown was somewhere other than by itself, on the right side of the equal sign. The errors were largely noncomputational and indicated that many children did not understand the meaning of the equation. The difficulty with symbolism seems to be pervasive and fundamental.

What happens when primary school children lack the understanding necessary to deal with arithmetic symbols in a meaningful way? The available research suggests that they begin developing their own system of rules to manipulate the symbols and generate answers. Many times they memorize fragments of algorithms or rules and recombine these in unique ways (Davis and McKnight, 1979). Lankford (1972) and Erlwanger (1975) have shown how intricate and "creative" some of these idiosyncratic systems can be. On occasion children's invented, incorrect algorithms are more complex than the correct ones. Apparently children are not incapable of learning complex algorithms and executing them consistently. In fact, Brown and Burton (1978) conclude that even when making errors, elementary school children are generally consistent and systematic. Frequently their errors are the result of methodically following the wrong procedure rather than making random mistakes.

What seems to be missing is a link with reality which might serve as a validating or correcting mechanism. Unable to make sense of the symbol system, many children appear to have no way of knowing whether the processes they are using are correct. While they may be convinced that the procedures they apply are the right ones, this confidence often comes from the belief that they have mastered the rules and tricks of the system, rather than a feeling that the procedures reflect reality (Erlwanger, 1975). A symptom of this problem is the periodic unreasonable responses provided by many elementary school children (see the report of the National Assessment of Educational Progress (NAEP) in Chapter II). Apparently, the formal symbolism of mathematics moves children from their natural, intuitive problem-solving skills that were anchored in real world experiences to rules of symbol manipulation, some of which have lost touch with their reality.

To reiterate using Ginsburg's (1977a) terms, children experience great difficulty translating their informal, experience-rich system into the formal, symbols-and-rules system of school arithmetic. Gaps between these systems begin to develop. Young children often are unable to establish meaningful links between what they know when they get to school and what they soon are asked to do—formalize this knowledge using mathematical symbols. As children lose their intuitive understanding of mathematical problems, or are asked to do mathematics in situations in which they cannot access these intuitive understandings, they begin developing their own unique systems of symbol manipulation, some of which are filled with misconceptions and faulty procedures.

## Implications for the Curriculum

The basic assumption of this chapter is that the way in which children think about mathematics and the processes they use to solve mathematical problems must be understood and taken into account by teachers and curriculum builders who design instruction If instruction is going to build on children's existing knowledge and the problem-solving strategies they have already developed, teachers must be aware of how children think about mathematics. However, an important note of caution should be inserted before discussing the implications of this research for mathematics curriculum and instruction. Research on children's thinking is necessarily *descriptive;* it describes children's behavior in learning or problem-solving situations. Instruction programs are essentially *prescriptive;* they prescribe the conditions that should be set up to facilitate learning. The prescription of programs does not immediately follow from the description of children's thinking (Bruner, 1966; Rohwer, 1970). Children's performance is the outgrowth of their learning experiences, and it is not always clear how a change in these experiences (through a change in curriculum) would influence this performance. Therefore, it is not always possible to prescribe the "best" curriculum from information on children's thinking. But it is possible to suggest several features that can be part of any instructional program.

Although there are many implications that might be drawn from the preceding review of research, two major implications stand out from the rest. One is suggested initially by Piaget's work and deals with the importance of observing children and looking at the world through their eyes. The second grows out of the recent work on children's mathematical thinking and centers on the importance of maintaining a link between children's natural base of experience and the mathematical concepts and symbols they are attempting to learn.

## Listening to Children

Children do not think like adults. They view the world from a different perspective; they solve problems by applying qualitatively different forms of thought. A striking example of this is young children's failure on conservation tasks, tasks that seem so "logical" to adults. But the difference in logic does not stop here. The research just reviewed suggests that the failure to understand conservation does not interfere with children's performance on mathematical tasks which, from an adult perspective, seem to depend upon this ability. Nonconservers successfully complete tasks for which conservation seems to be a logical prerequisite. It is clear that children think differently than adults.

Many researchers have pointed out the differences between children and adults, but it was Piaget who most clearly and profoundly demonstrated the nature of these differences. The success of Piaget's work can be attributed in part to his method of research. Piaget observed children as they were solving tasks, questioned them about the reasons for their responses, and tried to understand how they were thinking about the problems. Piaget's work, together with most of the research reviewed in this chapter, makes it clear that children's thinking can only be described by observing children as they are solving tasks. It is difficult, if not impossible, to describe or predict children's thinking by carrying out a rational, adult analysis of the task to be solved. The individual interview used by Piaget (Opper, 1977) suggests itself as a productive way to find out how well children understand basic mathematical concepts and to identify the processes they use to solve mathematical tasks.

The importance of this for instruction is that classroom teachers could apply these individual interview techniques with great benefit. Rather than administering only written tasks, teachers could schedule brief interviews with individual children and observe their performances on a few well-chosen problems. Questions can be asked in an accepting, nonevaluative way to uncover the processes used to solve problems. After identifying these processes, teachers can provide feedback on the appropriateness of solution strategies as well as the correctness of responses. Often children's existing strategies can be modified and built upon to create meaningful, appropriate strategies. Case (1978a) has described the importance of demonstrating to children any inadequacies of their current strategies and guiding them in acquiring more appropriate and efficient ones. In addition, information on children's processes provides teachers with a more fundamental understanding of children's errors.

Research has shown that children's errors are often the result of basic misconceptions rather than random carelessness. The nature of these misconceptions are difficult to diagnose by studying the responses on a paper-and-pencil test. Knowledge of the processes children use provides a deeper level diagnosis of their errors and provides a sound basis from which to prescribe appropriate instructional activities (Romberg, 1977).

## Developing Meaning for Mathematical Concepts and Symbols

In 1949, Van Engen pointed out the importance of relating the meaning of real world experiences with the arithmetic concepts and symbols that represent those actions or events. Instruction that emphasizes building these relationships is needed just as much today as it was three decades ago. The only difference is that a little more is known today about the informal knowledge upon which these relationships must be based, and the types of errors which result when the relationships are not built successfully. The implications described below emerge from this recent research base, but their intention could still be summarized by Van Engen's (1949) now-classic call for meaningful arithmetic instruction.

**Initial Instruction on Arithmetic Operations.** The research on early number and arithmetic concepts indicates that counting processes are critical in children's initial learning. Although all children enter school with some counting skills, not all of them are aware of the more advanced and efficient forms of counting. Rather than having children abandon their natural counting strategies, which have meaning for them, it may be beneficial to include material at the first-grade level directed toward improving these strategies. Children might be shown how to count forward from a given number, count back, and use various heuristic strategies to solve addition and subtraction problems. Two kinds of heuristic strategies that good counters seem to acquire naturally are those which use 10 as an intermediate number fact and those which use doubles (Carpenter and others, 1981a). For example, to solve the problem represented by $6 + 7 = \square$, some children will reason "$6 + 4$ is 10, and 3 more is 13"; other children will say "$6 + 6$ is 12, so $6 + 7$ is 13." These strategies suggest themselves as likely candidates for instructional content on counting.

An alternative to direct instruction on specific strategies is to provide opportunities for children to develop their own solution processes. Resnick (1980) proposes that instruction should be designed to put learners in the best position to invent or discover appropriate strategies for themselves. There is some evidence that even young children can invent strategies that are more sophisticated than those being taught if they understand the problems and are provided with appropriate aids for solving them (Groen and Resnick, 1977). Therefore, it appears that children may benefit from instructional methods that provide opportunities for them to develop and apply a variety of solution strategies. Strategies that children invent are likely to be strategies that have meaning for them.

Regardless of which approach is used for helping children develop efficient processes for solving addition and subtraction problems, it appears that verbal problems may be a good context in which to introduce these operations. It is frequently assumed that children must first master computational skills before they can apply them to solve problems. However, children develop a variety of counting strategies for solving verbal arithmetic problems before they receive instruction. This suggests that, rather than depending on prior knowledge of computation skills, these problem situations may give meaning to the basic arithmetic operations. In fact, verbal problems may be the most appropriate context in which to introduce addition and subtraction operations. Verbal problems also provide for different interpretations of addition and subtraction, interpretations which children bring with them to the school setting, and which must be eventually integrated for a full understanding of the basic operations.

**Initial Instruction on Symbolic Representations.** While many first-grade children are quite proficient at solving verbal problems, they often experience substantial difficulty dealing with symbolic expressions. It may be better to postpone formal symbolization until children have had a wide range of experiences with verbal problems and concrete arithmetic situations. There is some evidence which suggests that young children are able to use informal symbol systems (for example, tally marks, pictorial representations) to help them solve problems (Allardice, 1977; Kennedy, 1977). Perhaps these kinds of symbols would provide a more meaningful transition than now exists between children's intuitive understandings of the operations and arithmetic number sentences like $5 - 3 = \square$.

It is clear that when children first encounter the formal symbols of arithmetic, they have a difficult time developing meaning for the symbolic expressions. The emphasis during this instructional period should be on establishing and maintaining a link between the concepts children have already acquired and the symbols that are being introduced. Because of children's well-developed informal knowledge of verbal problem situations, these may provide the best context in which to introduce arithmetic symbols. The meaning children associate with verbal problems could be related to the number sentences that most directly represent the problem situation. To be successful, teachers will need to do more than simply present a related verbal problem alongside the number sentence (Grouws, 1972). Children need a variety of experiences in writing number sentences that represent verbal problems, and writing verbal problems which give meaning to number sentences.

Concrete materials can be used to represent arithmetic concepts and symbols physically. However, to help children see the connection between the physical representation and the symbolic representation, teachers need to structure the concrete activities so that frequent links are made between the physical and symbolic representations. For example, when using base ten blocks in the addition algorithm, symbols should be recorded immeiately after the objects have been manipulated in each column (Bell and others, 1976; Merseth, 1978); otherwise the concrete procedure functions as a calculating device which provides the correct answer but which does not facilitate a better understanding of the symbolic process. It is not just the use of concrete materials that improves mathematical understanding, but the explicit construction of a link between meaningful actions on the objects and the related symbol procedures.

In conclusion, the study of children's thinking provides some valuable insights into the processes children use to deal with mathematical situations, and generates some important suggestions for improving instruction. While the focus of this chapter has been on young children's thinking, the implications derived from research are equally appropriate for older children. Understanding what students are thinking as they solve long-division problems, or add two fractions with unlike denominators, is essential for developing instructional activities to help correct students' difficulties. Individual interviews, in which the student is asked to solve a few key tasks, and the teacher asks questions in a noninstructive, nonevaluative manner to clarify the solution procedures, could be employed to uncover the student's processes. Developing meaning for symbols is another instructional task that is equally important at all levels of mathematics learning. For example, teachers might help other children understand the difficult notion that two different fractional symbols can be equivalent (such as 1/2 and 2/4) by tying the symbols directly to their concrete and pictorial representations (Ellerbruch and Payne, 1978). Listening to students, and helping them to connect their meaningful base of experience to the symbols of mathematics are critical instructional strategies that can be used with benefit throughout the mathematics curriculum.

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# Response

Karen C. Fuson

The Hiebert chapter has identified and described a number of areas of current research on children's thinking. The short space available here permits the discussion of only two of the issues raised in the paper: first, a brief comment on the complex relationship between understanding and skill, and, second, some additional current research findings on ways in which the thinking of young children differs from that of adults and some suggestions to teachers about how to deal with these differences.

Gelman's proposal that preschool children come to understand counting principles and then gradually eliminate their execution errors in counting raises an old educational debate: whether understanding or rote performance comes first. In counting, as probably in other areas, the resolution of this debate is that each comes first in different aspects of counting. For example, the combined research findings of Gelman and Gallistel (1978), Fuson and Mierkiewicz (1980), and Mierkiewicz and Siegler (1980) suggest that children recognize that skipping an object in counting is an error before they are able to eliminate all of their own object skipping (that is, understanding precedes correct performance). On the other hand, while they rarely say an extra word when pointing to an object, many children do not think that doing so is a counting error (correct performance precedes understanding). Thus, there may be some things that are easier for children to understand than to do and others that are easier simply to do than to understand.

One of the differences between the thinking of children in kindergarten through second (and even perhaps third) grade and that of adults is that children are not nearly as capable as adults are of planning and organizing their activities. That is, there are not only cognitive differences between children and adults; there are also meta-cognitive ones. Children's actions are much more dictated by things or people perceptually present in their immediate environment than by specific goals acting over a relatively long period of time. Children of this age are capable of goal-directed activity, but they also tend to spend much of their time reacting to immediate stimuli in their environment rather than enacting pre-set plans.

Thus, two major functions of a teacher of young children are (1) to provide activities with goals of relatively short duration so that children can maintain their activity in a goal-directed way, and (2) to construct a learning environment containing stimuli to which children can react. The latter suggests the use of concrete objects and concrete situations. Such use has long been proposed for cognitive reasons: Piaget's legacy has been to help us realize not only how concretely bound children's thinking is, but also what capable thinkers young children can be when they are provided with the concrete tools they require for thinking. However, another benefit of the use of concrete objects and situations with children is now evident: a meta-cognitive one. Such perceptually present stimuli in the immediate environment will serve to organize children's behavior and keep it focused on the desired stimuli for learning.

The theoretical work of Vygotsky (1962, 1978; Fuson, 1980) has focused renewed interest on adult-child (or teacher-learner) interaction in learning. This interaction in the learning process can be viewed as one in which the adult and the child engage in a common goal-directed activity with the adult at first carrying out many of the parts of this activity. Gradually the child learns to do these parts and the adult becomes less active, limited to organizing, monitoring, and supporting functions. Finally even these functions are eliminated, and the child takes over the whole cycle of behavior. The Soviets call this the movement from the inter-psychological to the intra-psychological plane.

I have outlined and discussed an application of this notion to the mathematics classroom (Fuson, 1979). One of the purposes of such a model is to create a meta-cognitive change in the teacher: to help the teacher step back from the busy acting and reacting that occurs from minute to minute in the classroom and to reflect on these meta-processes inherent in the teaching-learning process. In particular, the goal is to help the teacher think about how to help move children from being nonplanning reactors to actors actively involved in reflecting on and organizing their own learning. Young children do not spontaneously *reflect upon* their actions and thoughts. They can do so, however, with the help or suggestion of the teacher. Repeated help and suggestion will then serve to increase spontaneous reflection.

One way to accomplish this transfer of organization is by using verbalization. Procedures can be verbalized by the teacher; students can then use these verbalizations to regulate their own behavior when they are carrying out the same procedure. This occurs spontaneously in the classroom now: one sees lips moving as a child verbalizes his or her way through an addition solution.

("Write down the four and carry over the one to the tens place.") However, such verbalizations could be used deliberately by teachers.

Children's thinking also differs from that of adults because there is a great deal of information children simply do not know. Thus, one of the primary and obvious functions of a teacher is to help children learn some of the many facts about their world that can then also enhance their reasoning processes.

A related teaching function that is less obvious is to help children recall relevant information they may possess but may not recognize is relevant. Children's knowledge is initially quite context-bound: what they learn is related to the specific context in which they learned it. This is further complicated by the fact that children often notice, encode, and remember surface features of a problem situation rather than the underlying structural dimensions that would be processed by older children and adults. For example, the Soviet researcher Krutetskii (1976) discovered that when children who were very good at mathematics were asked about some story problems they had solved on an earlier occasion, they had encoded and remembered the operation involved in the story. (For example, "Oh, there was one about a boy having some things and another boy giving him some more of them.") Children who were not so good at mathematics had encoded and remembered mathematically irrelevant features of the stories. ("There was one about trains.")

This difference in the encoding of a situation has now also been found by American researchers, who consider it to be one of the major differences distinguishing experts from novices in a field. Expert chess players, physicists, and so forth notice underlying structural features of situations that novices do not notice. Because primary school children tend to be novices at practically everything, teachers should keep in mind the fact that children are likely to focus on attributes of a situation different from those noted by the expert teacher.

A very important part of the teacher's role, then, is to help children begin to notice structurally important features of a situation (such as numbers of objects) rather than more obvious surface features (such as the color of the objects). Similarly, teachers will explicitly need to point out (or help children discover for themselves) how certain contexts are similar to other contexts the children have experienced, but which they may not recognize are "the same" in some important way. (For instance, 5 + 2 yields the same result as 2 + 5, though the person who started with 2 and got 5 may be much happier than the one who started with 5 and got 2.)

A final difference between the thinking of young school children and adults is that children do not have the same mental operating capacity that adults have, and they cannot assess information as quickly as adults can. For example, in a task in which one is supposed to count the number of objects on a card while remembering the number of objects on previously counted cards, four-year-olds cannot remember the previous count, six-year-olds can remember the count of two cards, and adults can remember the counts of four cards (Case and others, 1979). This means that teachers have to figure out ways to help children use external memory (things written on paper, for instance) rather than overload their mental processes. One example of such an overload involves regrouping in the usual addition algorithm. In the problem 47 + 38, after the 1 is carried over to the tens column, the child must do two mental addition problems, holding the sum for the first one in mind as an addend for the second (1 plus 4 is five, five plus 3 is 8). An alternative procedure (Ames, 1975) using external memory is to add in the 1 to the 4, crossing out the 4 and writing 5 above it. The child then is presented with only the second of the two addition problems (5 plus 3) and does not have to remember anything.

The current overemphasis by researchers on counting to the exclusion of other sources of arithmetic ideas should not be emulated by teachers. The use of measure notions embodied by Cuisenaire rods (seven is a certain length) and of certain figural patterns that can be combined and separated to form sums and differences are also very important, especially for certain children. Providing a rich range of mathematical experiences from which children can choose those most consistent with their particular pattern of thinking is critical.[1] The mathematical world models many kinds of external realities, and the paths to understanding this world are themselves many and varied.

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[1] See, for example, Baratta-Lorton (1976); the Nuffield project books for early grades; and Bell, Fuson, and Lesh (1976) for both counting and measure materials for the later grades.

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# IV. Teachers' Decision Making

Thomas J. Cooney

Teachers are sometimes likened to actors on stage: they emote and enthuse in order to capture the imagination of the audience. While teachers may need to be good actors, the process of teaching also includes *re*acting. Teaching is an interactive process, one in which the teacher plays off the students and the students play off the teacher. It is a process of gathering information, making a diagnosis, and constructing a response based on that diagnosis. While much of this process may be quite automatic, some situations require conscious decision making. The act of generating and considering alternatives in constructing a response—that is, making an instructional decision—is of paramount importance in teaching.

Shroyer (1978) used the term "critical moments" to denote those moments of classroom teaching when there is an occlusion in the instructional flow. Perhaps a student demonstrates an unanticipated learning problem or gives a particularly insightful response. Such unexpected events cause the teacher to reflect on the interaction and to process certain information in order to construct a reaction. Episodes that depict critical moments are presented later in this chapter to provide a context for considering teaching as a process of decision making.

## The Decision-Making Process

Various researchers have studied teachers' decision-making processes (Shavelson, 1976; Peterson and Clark, 1978). Regardless of the theoretical

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The author would like to express his appreciation to Dr. Stephen I. Brown of the University of Buffalo for his helpful comments in writing this chapter.

prism through which the processes are viewed and studied, several aspects remain constant: teachers gather and encode information, generate alternatives, and select a course of action.

In Peterson and Clark's study (1978), a scheme consisting of four paths was developed for describing teachers' decision-making processes. The investigators found that Path 1 was most frequently traversed; Part 4 was the second most traversed; student achievement was negatively correlated with Path 3; and Path 4 was positively related to higher learning outcomes. Peterson and Clark's study emphasizes two aspects of teaching central to decision making: (1) the decision-making process is related to educational outcomes, and (2) a critical part of the decision-making process is the generation of alternatives. The generation of alternatives is considered central to viewing the teacher as a decision maker and is deemed essential for a flexible and creative teacher. Peterson and Clark's analysis helps provide a means by which we can consider the role alternatives play in the decision-making process.

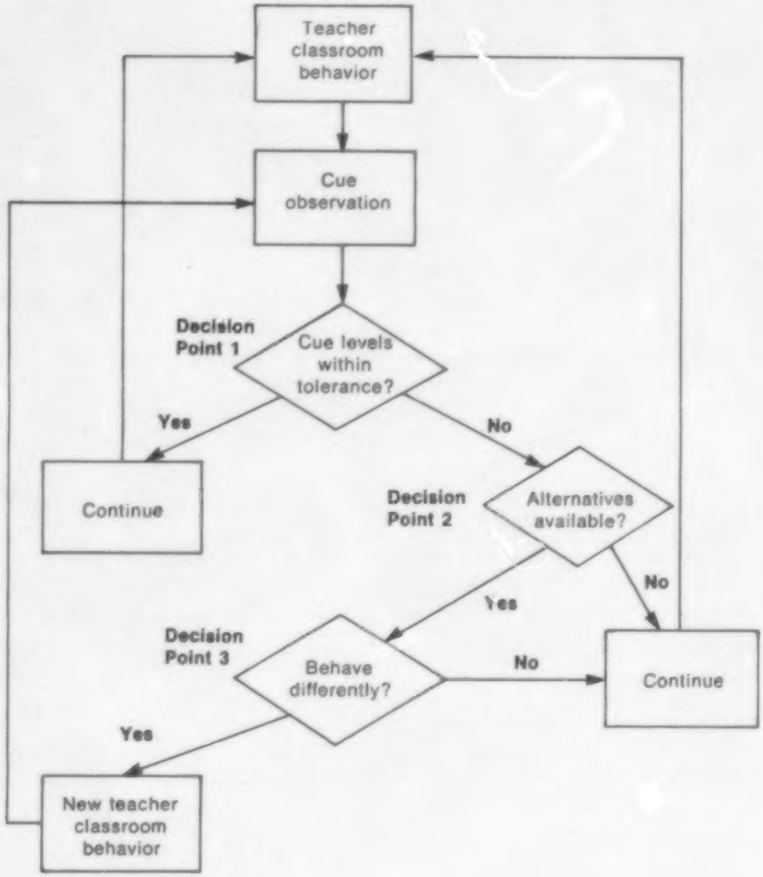
## Types of Decisions

Teachers make different types of decisions. Some are related to the content, including its selection, and the selection of teaching methods. Other decisions relate to the more interpersonal aspects of teaching, that is, affective concerns. Still other decisions involve management considerations, including the allocation of time. I will use this triadic scheme of classifying decisions as cognitive, affective, or managerial to focus on the various types of decisions that teachers make. I must emphasize, however, that these three categories are not in any way mutually exclusive. Teaching is too complex to permit such a simplistic view. In the real world of the classroom, classification schemes are seldom clearly exhibited. Nevertheless, the classification seems appropriate at least for the purpose of examining factors that influence decisions.

### Cognitive Decisions

There are two phases of teaching. The *preactive phase* is what transpires before the teacher begins interacting with students. It typically involves lesson planning. The *interactive phase* involves the classroom interaction between students and the teacher. Content related dcisions, as well as other types of decisions, are made in both the preactive and interactive phases.

**Figure IV-1. Scheme for Analyzing Decision Making by Teachers**


Teacher classroom behavior
Cue observation
Decision Point 1
Cue levels within tolerance?
Yes
No
Continue
Decision Point 2
Alternatives available?
No
Yes
Decision Point 3
Behave differently?
No
Continue
Yes
New teacher classroom behavior

**Paths Identified From the Scheme**

| Decision Points | Path 1 | Path 2 | Path 3 | Path 4 |
|---|---|---|---|---|
| *Student Behavior Within* | | | | |
| Tolerance? | Yes | No | No | No |
| Alternatives Available? | — | No | Yes | Yes |
| Behave Differently? | — | — | No | Yes |

A content decision that occurs in the preactive phase is deciding which content to present and which to exclude from the instructional program. Cooney, Davis, and Henderson (1975) identified the following factors that affect teachers' decisions in selecting content: (a) requirements or regulations from governing bodies, such as state departments of education, (b) objectives developed by a teacher, department, or a more inclusive group, (c) the expected use of the content to be taught, (d) the student's interest in the content as well as the teacher's interest in teaching it, (e) the predicted difficulty of the content, and (f) authoritative judgments expressed by professional groups or prestigious individauls within the field. In many cases decisions related to topic selection are passive and based primarily on what appears in textbooks. Nevertheless, a decision is made.

Another type of content decision concerns how the content within a topic will be interpreted or presented. Consider the concept of fraction. One can conceive of at least ten different interpretations of fraction: parts of a region, parts of a collection, points on the number line, fractions as quotients, fractions as decimals, repeated addition of a unit fraction, ratios, measurement, operators, and segments. Decisions must be made on which one or which combination of interpretations to use in teaching fractions. Similarly, there are various means of interpreting other mathematical topics. Such interpretations provide a variety of alternatives to consider when presenting content.

Decisions are also made with respect to strategies of presentation. A variety of materials, such as rods or paper folding, can be used to present different interpretations of the content. Another strategy decision has to do with the use of examples and nonexamples. Suppose the teacher wants to develop the concept of line symmetry for the class. A matrix similar to the one below could be constructed with students providing the samples.

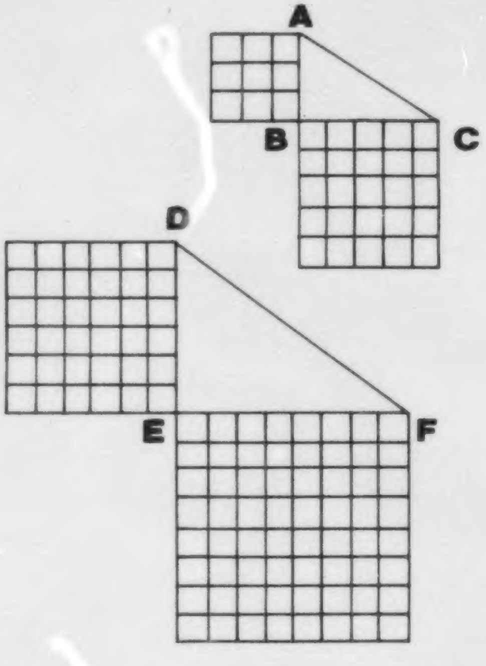
| | Mathematics | Real World Applications |
|---|---|---|
| Example | rectangle | a human face |
| Nonexample | parallelogram | a human hand |

Such an activity can provide a mixture of examples and nonexamples and relate the concept to life-like situations. Other cognitive decisions include deciding how to justify theorems, what prerequisite knowledge should be reviewed, or whether to use an expository or a discovery approach.

Cognitive decisions are also made in the interactive phase of teaching. Several classroom episodes are posed below to highlight the nature of these types of decisions.

### *Episode 1*

Mr. Smith's class is learning the Pythagorean Theorem. Students had used unit squares to construct larger squares on the legs of right triangles ABC and DEF.



These unit squares were then rearranged to form a larger square of unit squares on the hypotenuse. The following dialogue between the teacher and two students, Billy and Chuck, then transpired.

*Teacher:* Now consider the right triangle with legs of length a and b and hypotenuse c. *(He draws triangle ABC on the board.)* What does the theorem say about this triangle?

*Billy:* $a^2 + b^2 = c^2$.

*Teacher:* Okay. Very good. Now suppose we have a different right triangle with legs of length a and c and hypotenuse b. *(He draws this triangle on the board.)* Now what does the theorem say?

*Chuck:* The theorem won't work for that triangle. It doesn't apply.

Apparently, Chuck had not grasped the meaning of the theorem. Perhaps he thought of mathematics only in terms of symbols and not in terms of meanings behind the symbols. What alternative actions exist? Possibilities include the following:

a. Call on another student to state a relationship.

b. Tell Chuck the theorem does apply and state the correct response.

c. Ask Chuck to clarify what "doesn't apply" means.

d. Ask him to state the conditions under which the theorem does or does not apply.

e. Ask another student if he or she agrees.

The issue is not which alternative is necessarily better for all situations. Rather, the focus should be on the identification of possible alternatives and the decision as to which one seems best suited in a particular context. The making of a wise decision requires the consideration of various alternatives in light of what is known about a particular student in specific situations.

### *Episode 2*

Ms. Jones was reviewing linear functions when the following dialogue occurred.

*Teacher:* What do we mean, class, by linear function? How would we define it, Mary?

*Mary:* I don't know. I forgot.

*Teacher:* Carla?

*Carla:* Well, it has something to do with a straight line.

*Teacher:* That's true, but we need more.

Evidently Ms. Jones perceived that students were struggling with the apparent goal of stating a definition. At this point, several alternatives could be considered, including the following:

a. Call on another student and press for a correct definition.

b. Provide some sort of a hint on how to "start" the definition and give Carla or another student a chance to state the definition.

c. Abandon the instructional goal and identify a new goal.

The dialogue continued.

*Teacher:* Jan?

*Jan:* Things like $f(x) = 2x + 3$ and $f(x) = 4x - 10$. These are linear functions, aren't they?

*Teacher:* Yes. That's good. Okay, now let's see how we can graph some linear functions.

The teacher seemed satisfied with the two examples. Was she unclear about the content being taught or at least unclear over the distinction between definitions and examples? Did the teacher make a conscious decision to accept examples rather than a definition? If so, what factors influenced her decision? What was the likely impact of the discussion on the students? Were they confused about what constitutes a definition?

We cannot be sure what cues Ms. Jones attended to when she made her decision to accept the answer, or if she considered any other alternatives. In short, we do not know what information this teacher processed in making the decision. But we do know that for whatever reason a response which was not an answer to the teacher's question was finally accepted. The response may have been accepted as a compromise if Ms. Jones perceived that the task was harder than anticipated (and thus the goal was changed). Or it may have been accepted without Ms. Jones reflecting on the nature of the instructional request.

Consider another situation observed in an elementary school classroom.

### *Episode 3*

Mr. Costa's class was discussing the addition of whole numbers. At one point the discussion focused on a word problem that entailed finding the sum of 1970, 330, and 31. The following dialogue occurred.

*Teacher:* So what numbers do we need to add?

*Sonya:* 1970, 330, and 331.

*Teacher:* Okay. Albert, why don't you show us on the board how to add those numbers? *(Albert goes to the board and writes the following.)*

```
1 9 7 0
3 3 0
3 1
-------
8 3 7 0
```

*Albert:* The answer is 8370.

Albert's difficulty and misconception are clearly evident. What alternatives exist for the teacher?

a. Ask another student to come to the board and find the sum.

b. Show Albert and the class how the numerals should be arranged.

c. Use the idea of place value to explain briefly how the numbers should be added.

d. Stop the lesson to review in some detail the process of adding whole numbers.

In this particular case, the teacher decided on option b. The effect seemed to be a continuation of the class discussion in a fairly uninterrupted manner, although an observer might wonder if Albert's confusion had really been resolved.

The following episode highlights the importance of generating alternatives when a lesson goes poorly, and the importance of generating alternative strategies when planning a lesson.

### *Episode 4*

Mrs. Lincoln, a seventh-grade mathematics teacher, was teaching her class how to factor whole numbers into their prime factors. She began by quickly stating the definition of *prime number* and giving two examples of prime numbers. No nonexamples were given. She then presented two demonstrations of how to obtain the prime factorization of a whole number. Students had obvious difficulties, including the above mistakes, as alleged by prime factorizations of the numbers on the left.

$12 = 4 \times 3$
$8 = 5 + 3$
$40 = 4 \times 10$
$24 = 16 + 8$

Mrs. Lincoln recognized there was a problem; she repeated the definition, and gave one more demonstration. Students returned to their worksheets, but few corrections were made as they were still quite confused.

Several comments are relevant. First, the students lacked basic prerequisite knowledge with respect to the concepts of prime number and factor. Had the teacher placed greater emphasis on teaching these concepts, particularly through the use of examples and nonexamples of prime numbers and by comparing factors with addends, students would likely have done better.

Second, it seems clear that Mrs. Lincoln had few instructional alternatives to draw on. The role examples and nonexamples can play in designing instructional strategies was mentioned earlier. Kolb (1977) developed a model for predicting the effect of various strategies, including the use of examples and nonexamples for teaching mathematical concepts.

Basically, Kolb's model suggests that examples and nonexamples of concepts produce more learning than presenting characteristics of concepts when students have little prerequisite knowledge. For students with a higher degree of prerequisite knowledge, discussions that focus more on the attributes of a concept, for example, necessary and/or sufficient conditions for concepts, are more effective than focusing on specific examples and nonexamples. The model is complex and involves considerable detail. However, it does highlight the importance of using examples and nonexamples, particularly for students with poor conceptual backgrounds.

In Episode 4, it was clear that many students did not understand the concept of prime number nor of factor. For them an instructional alternative should have been generated which entailed extensive use of examples and nonexamples. Interation of the strategy "define and give one or two examples" was not productive.

## Affective Decisions

Teachers need to be sensitive to students and provide ample affective support for them. Instructional decisions involving affective considerations are sometimes based on the teacher's perception of how students are interacting with the content. Comments like "Why do I have to learn this?" are not atypical in mathematics classrooms. The way in which such questions are handled depends on what the teacher perceives to be the reason for such a comment. If the student is asking "How does this content fit with other topics that we have studied or will study?" or "How can the content be applied to help me solve problems in the real world?" then a response dealing with the substance of the discipline is appropriate and, hence, is primarily cognitive in nature. But if the student is really asking "Why am I not doing better in learning this?" then a response oriented toward building the student's confidence appears more appropriate. Thus a teacher is faced with an instructional decision. Within the affective domain in particular, hidden meanings must be attended to as well as the overt context of the remark in order to generate viable alternatives.

Bishop and Whitfield (1972, p. 35) offered the following situation, which suggests the need for an "affective" response:

> If a man can run a mile in four minutes, how far can he run in an hour? A 12-year-old pupil answers: "Fifteen miles." On being questioned about the reasonableness of the answer, he replies: "Well, math is nothing to do with real life, is it?"

Should one expect that a substantive discussion on the relationship or applicability of mathematics to the "real world" would resolve the problem? Perhaps, but it is also conceivable that the student's response has less to do with mathematics per se than it does with an affective problem associated with learning mathematics.

Consider the following two episodes.

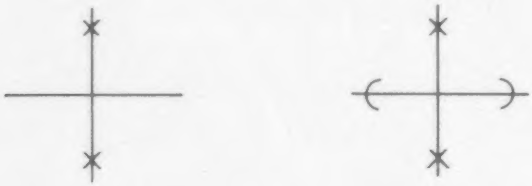
### *Episode 5*

Donald has considerable trouble learning mathematics. The current lesson is on solving linear equations of the form $ax + b = c$. Donald is doing rather poorly. The teacher has emphasized that it is important in solving equations to have only one equal sign (=) per line. Donald, along with other students, is sent to the chalkboard to practice solving equations. Donald typically does not do well when performing at the board. The equation to be solved is $2x + 4 = 7$. Donald's solution is:

$$2x + 4 = 7 = 2x + 4 - 4 = 7 - 4 = 2x = 3 = x = 1\tfrac{1}{2}$$

*Episode 6*

Pat is a C student in geometry. The class has been studying constructions using a compass and a straightedge. Most students are quite proficient in bisecting a line segment as shown on the left. However, Pat persists in bisecting a segment in the manner indicated on the right.



The teacher has continually emphasized to Pat that while her procedure is mathematically correct, it is not the most efficient way and not the method to be used in class. Nevertheless, when asked to find the midpoints of the sides of a triangle, Pat resorts to the second method.

If one were to consider only affective concerns to the exclusion of cognitive ones, then decisions would be easier. But often affective decisions must be tempered with cognitive concerns, as evidenced in Episodes 5 and 6.

In Episode 5, the teacher reinforced Donald with considerable praise for obtaining the correct answer. As a result, Donald felt proud but other students asked if they could solve equations using only "one" line. The teacher seemed intent on emphasizing affective outcomes; desirable affective outcomes were paramount to the teacher. In Pat's case, the teacher was very sharp and critical. Pat probably wouldn't make the same "mistake" again, but at the expense of a loss of enthusiasm for the subject. For this teacher and this situation, cognitive outcomes were evidently of higher priority than affective outcomes.

Many decisions involve striking a balance between cognitive and affective outcomes. Recall Chuck's response concerning the Pythagorean theorem. Chuck had a misconception regarding the theorem. But the teacher might select an alternative action having considerable affective overtones. That is, an alternative might be selected which best ensures Chuck's feelings would not be hurt or best ensures Chuck's continued participation in class discussions. This situation highlights the necessity of considering a number of factors, both cognitive and affective in nature, when making instructional decisions. Artistic teachers are often able to promote both desirable cognitive and affective outcomes. One type of outcome need not be sacrificed for another. But the task of striking a balance is not always easy; it requires careful consideration of several alternatives of action.

Research generally indicates that the warm, supportive teacher is more effective than the critical teacher. Tikunoff and others (1975) conducted an ethnographic study of second- and fifth-grade teachers teaching reading and mathematics in which many teaching variables, affective in nature, were found to be related to achievement. The investigators characterized the significant variables as being related to "those familial interactions in the home which have been attributed traditionally to the successful rearing of children" (p. 22).

Rosenshine and Furst (1971) also suggest that the warm, supportive teacher is more effective than the critical teacher. However, Brophy and Evertson (1976) found that in high socioeconomic status (SES) classrooms praise was negatively related to student learning gains, whereas students in low SES classrooms prospered in warm, supportive classroom atmospheres. This suggests that affective variables may be contextual in nature in terms of how they relate to achievement.

Teachers make continual assessments of students' affective status in the classroom. Although universal quantification is difficult to justify, generally the "familial" variables identified by Tikunoff and others (1975) seem to characterize the effective teacher. But individual instructional decisions may not be unidimensional in value. That is, one may have to strike a balance between cognitive concerns and affective ones when assessing expected payoffs of various teaching behaviors.

## Managerial Decisions

Managerial decisions relate to time allocation, organization of classroom activities, and control of disruptive behavior. Some of these decisions can be made in the preactive phase of teaching while others, especially those related to "control" problems, are more specific to the interactive phase of teaching.

Consider Episodes 7 and 8, which involve decisions related to how time is allocated.

### *Episode 7*

A student is subtracting fractions and keeps making mistakes similar to the one below.

$$\begin{array}{rcl} 4\ 1/8 & = & 3\ 11/8 \\ \underline{-1\ 7/8} & = & \underline{1\ \ \ 7/8} \\ & & 2\ \ \ 4/8 = 2\tfrac{1}{2} \end{array}$$

After the teacher poses several questions, it is clear the student is quite confused.

### *Episode 8*

A geometry teacher is discussing the importance of the parallel postulate in Euclidean geometry. The teacher has emphasized that many theorems in their geometry books are based on the parallel postulate. As an illustration, the teacher argues that the theorem, "The sum of the measures of the angles of a triangle is 180 degrees," follows from the parallel postulate. A bright student asks, "If we didn't have the parallel postulate, does that mean the measures of the angles of a triangle would be different than 180 degrees?"

In Episode 7, should content be reviewed for a single student or for a few students at the risk of "wasting" the time of other students? In Episode 8, should class time be taken to pursue the thought initiated by the bright student? Or should the student be informed that the question was a good one and it would be followed up sometime *after* class? What are the expected results of the two alternatives? The decisions will clearly affect how time is allocated. What is not so clear and is quite value laden is deciding how to strike a balance between discussions of a tangential point for a few students compared with discussions that benefit the remaining students. Given that instructional time is a scarce commodity, allocation of that commodity is critical to determining what is learned.

Some decisions on time allocation occur in the preactive phase of teaching. Ebmeier and Good (1979) found that fourth-grade mathematics teachers could improve achievement by emphasizing six aspects of instruction with tentative time allocations: development (about 20 minutes), homework, emphasis on product questions, seatwork (10 to 15 minutes per day for practice), review/maintenance, and pace (consider the rate of instruction and increase if possible).

Berliner (1978) reported a great deal of variance among teachers in how they allocate time for mathematics instruction, particularly for specific topics, such as fractions, measurement, decimals, or geometry. At the elementary level, the time allocated for mathematics instruction varies considerably from one day to the next because of contextual situations, for instance, if students come back late from a music class or a social studies project takes longer than expected. At the secondary level, the allocated time is more constant, but even within that allocation, a teacher may decide to take care of administrative tasks or attend to other nonmathematical activities. Thus, a decision of one sort or another may significantly affect the amount of time devoted to the study of mathematics.

Another type of decision, which occurs in the interactive phase of teaching, is the decision on how long to wait for students to respond to a question. Rowe (1978) defined two kinds of wait time: (1) the pause following a question by the teacher and (2) the pause following a student's response (usually measured in terms of seconds).

Rowe (1978) found that elementary science teachers typically wait less than one second before commenting on an answer or before asking an additional question. When the two types of wait time were increased, Rowe reported that the length of student responses increased, failures to respond decreased, students' confidence increased, disciplinary problems decreased, slower students participated more and, in general, students were more reflective in their responses.

Consider the likely payoff if wait time of less than one second predominates. Can problem-solving abilities be nurtured and promoted when wait time is consistently less than one second? Not likely. It seems highly desirable for teachers to be explicitly aware of concepts such as "wait time" in order that alternatives can be generated which are consistent with their instructional goals. This is not to claim that awareness of such concepts will yield completely "rational" decisions in the sense that an explicit and highly recognizable decision-making strategy can be readily identified. But it is the belief here that whatever commonsensical decisions are made in the classroom, they can be enhanced by an explicit awareness of alternatives and by having a variety of pedagogical concepts, of which wait time is one, on tap.

Another aspect of managerial decisions involves the ever-present problem of discipline. To deny that teachers are concerned and conscious of potential and actual classroom disruptions is to be oblivious to the realities of classroom teaching. Consider the following episode.

### *Episode 9*

A first-year geometry teacher was discussing the proof of a theorem with the class. In the back of the room a student who was the band's drum major was twirling her baton. After a minute or so, the young teacher noticed her behavior. The teacher's confusion about alternatives was mirrored on his face. Apparently, alternatives did not exist since the teacher avoided the situation. But the impact on the class of the indecisiveness could not be discounted.

Perhaps the response of "do nothing" was the best alternative. But consider an alternative prior to the specific incident. Could the teacher have moved about the room (as was not the actual case) and, as a result, increased his awareness of any potential problems? Did not the decision, determined consciously or unconsciously, to stay in front of the class in a small area inhibit his ability to monitor the student's behavior in the back of the class? Had he decided to move around the room and consciously monitor student behavior, could the embarrassing incident have been avoided? Probably so.

The ability of a teacher to monitor classroom events has been the focus of various investigations. For example, Kounin (1970) studied a number of variables with respect to classroom management and their relationship to achievement. One of the variables identified was called "withitness." This variable dealt with teachers communicating that they know what is going on regarding children's behavior and with their ability to attend to two issues simultaneously. Kounin found withitness to be a strong correlate of achievement. Brophy and Evertson (1976) also found that more successful teachers were more "withit" than less effective teachers. Thus it appears that a teacher's ability to monitor simultaneous classroom events is an important factor in maintaining control and in positively affecting achievement.

There are no explicit directions for solving management problems. But alternatives can be identified for preventing and coping with situations. Perhaps an explicit awareness of possible alternatives can assist teachers in making those difficult decisions and provide greater confidence in themselves for believing they can control classroom events.

## Conclusion

Teachers have an immense amount of common sense and good judgment. Many creative teachers have a wealth of alternative methods for dealing with a wide variety of classroom situations. But common sense can be enhanced by an explicit awareness of the importance of generating alternatives and by an explicit knowledge of various pedagogical concepts and principles. Practitioners' maxims and research in concert can play an important role in the generation of alternatives. The art of teaching can be improved by consciously considering alternatives and by expanding the knowledge base for generating alternatives.

Another aspect of improvement can arise from reflecting on why certain alternatives are selected. Value judgments, perceptions about what constitutes the teacher's role, and what constitutes mathematics all provide a sort of filter through which some alternatives pass and others do not. Perhaps a realization of what factors contribute to the selection of alternatives as well as an awareness of the decision-making process itself can provide a basis for several outcomes: additional insights into the teaching process, a richer use of the teacher's knowledge base, and an avenue for teachers' further professional development.

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# V. Process-Product Research

Thomas L. Good and Douglas A. Grouws

The 1970s produced a considerable amount of substantive research on the teaching of mathematics. Different researchers chose to study the complex processes and interactions associated with mathematics teaching using varied methods. These research approaches may be broadly classified as either qualitative or quantitative. Qualitative approaches include ethnographic studies that draw heavily on the methods typically used by anthropologists and focus on questions concerning what is happening in a classroom and why it is happening. The research on general mathematics teaching by Confrey and Lanier (in press) as well as the research by Easley and his associates (1977) have profitably employed qualitative methodology. Studying mathematics teaching via information-processing approaches also seems to hold potential. Resnick is doing some promising work using this methodology (Resnick and Ford, 1980).

The quantitative approach to studying the teaching of mathematics includes aptitude-treatment interaction studies (Which instructional treatment is best for which student?), process-product studies, and the more traditional treatment-control group type studies (where usually only a few instructional variables are manipulated). The goal of the quantitative studies is to identify what works for specific groups of teachers or specific groups of learners (see, for example, Janicki and Peterson, in press). There is less focus on an individual student or teacher, and more emphasis on groups of learners or teachers.

Both qualitative and quantitative research endeavors are beginning to provide some clear implications for instruction in mathematics. These implications are clear in the sense that they seem logically sound and have also withstood the test of replication or identification in more than one setting. In writing this chapter, we have been asked to focus on the process-product approach to mathematics research and the recent experimental work it has stimulated. This is only a small part of the research being done on the teaching of mathematics, and we are aware of the valuable insights that are being produced by other approaches (see, for example, Davis and McKnight, 1979).

## Process-Product Studies

An area that has shown considerable promise in recent years is the study of teacher behaviors. Many of the behaviors that have been associated with effective teaching have been identified from what are commonly called process-product studies. In this type of research, a set of teacher behaviors that seems to hold potential for producing student learning are identified and defined. The frequency and extent of their occurrence are then determined in many classrooms over a fixed period of time. Finally, the correlation between the frequency of occurrence of these teacher behaviors and the average class achievement scores (adjusted for initial differences) during the observation period is computed. A high positive correlation between one of the behaviors and mathematics achievement suggests that effective teachers use this instructional strategy or behavior more often. Replication of the teacher behavior-pupil achievement relationship in subsequent naturalistic studies gives credibility to the finding and suggests the need to examine the variable in field-based experimental studies where cause and effect relationships can be assessed more adequately.

During the past few years a large number of process-product studies of teaching have been conducted (for review, see Brophy, 1979; Good, 1979; Peterson and Walberg, 1979). Several of these studies have specifically examined the teaching of mathematics. In a study of fourth-grade mathematics instruction, Good and Grouws (1977) identified nine effective and nine less effective teachers from a sample of over 100 teachers. Over a three-year period the effective teachers consistently produced better-than-expected mathematics achievement results (residualized gain scores), while the less effective teachers consistently produced lower-than-predicted achievement gains. These differences in outcomes occurred despite the fact that the students taught by relatively effective and ineffective teachers were comparable in ability.

Observational data were collected in 41 classrooms to protect the identity of the relatively effective and ineffective teachers. Approximately equal numbers of observations were made in all classrooms (6-7). Data were collected by two trained observers (both certified teachers) who worked full-time and lived in the target city. Each coder visited all 41 teachers and made about half of the observations obtained in a given classroom. Furthermore, all observations were made without knowledge of the teacher's level of effectiveness.

The data from this study demonstrate an important fact, that patterns of consistent behavior can be identified for both high-effective and less-effective mathematics teachers. Further, there are differences in the patterns for the two groups and these differences suggest behaviors associated with effective and less effective instruction in mathematics, as measured by standardized achievement tests.

Before examining these differential behaviors, two points need to be made. First, many of the teachers in the study did not perform in a consistant fashion. One year they might obtain very good results, and the next year their students might achieve far less than expected. The reasons for these fluctuations are not known and have not been studied. They do suggest, however, that subtle context factors may influence teacher effectiveness (Good, 1979). For example, it may be that if the variability of student ability within a class exceeds a critical point, then the teacher is not successful, even though he or she is behaving and interacting in exactly the ways that had previously produced good results. However, teachers who have inconsistent effects might also vary their behavior from year to year as events in their personal lives allow them more or less time for teaching.

It is highly unlikely that any given behavior in isolation is going to profoundly affect achievement or determine who is an effective teacher. It is far more likely that a number of interrelated behaviors simultaneously (probably under specific conditions) stimulate and enhance student learning in mathematics. Because of the large number of correlations examined in process-product studies, it is also possible that some of the behaviors identified as being associated with effective teaching are not valid. For these reasons it is particularly important in analyzing and interpreting process-product data to look for *clusters* of related behaviors that seem to be associated with effective teaching. With this perspective in mind, let us examine some results.

In the study of fourth-grade mathematics, teacher effectiveness was found to be strongly associated with the following behavioral clusters: (1) general clarity of instruction; (2) task-focused environment; (3) nonevaluative (comparatively little use of praise and criticism) and relatively relaxed learning environment; (4) higher achievement expectations (more homework, faster pace); (5) relatively few behavioral problems; and (6) the class taught as a unit. Teachers who obtained good results were very active (that is, they demonstrated alternative approaches for responding to problems), emphasized the *meaning* of mathematical concepts, and built systematic review procedures into their instructional plans.

In another process-product study focusing on mathematics, Evertson and others (1980) carefully selected a small sample of effective and less effective seventh- and eighth-grade teachers and then systematically observed their teaching. They found that more effective teachers, in contrast to less effective teachers: (1) spent more time on content presentations and discussions and less time on individual seatwork; (2) held higher expectations for their students (assigned homework more frequently, stated concern for academic achievement, and gave academic encouragement more often); and (3) exhibited stronger management skills (minimized inappropriate behavior, made more efficient transitions, and had more student attentiveness).

A number of important relationships exist between the findings of these two studies and earlier research. The conclusion regarding the value of time spent on content presentations coincides very closely with the results of a large number of experimental studies in mathematics (for example, Schuster and Pigge, 1965; Shipp and Deer, 1960; Zahn, 1966; and Dubriel, 1977). These studies have specifically examined the development-practice variable and found without exception that spending more than half of the class period on developing skills and ideas results in higher student achievement. The importance of teacher behaviors that communicate high-achievement expectations in several studies is significant. Similarly, the finding that strong managerial skills and few behavioral problems are positively associated with student achievement seems to be a significant link between the two studies and earlier research (Kounin, 1970).

In a process-product study of the teaching of ninth-grade algebra, Smith (1977) found three interrelated teacher behaviors associated with pupil achievement gains. Smith concluded that these behaviors could probably be associated with a more global variable "involving organization, structuring and clarity of lessons." Here again are findings very similar to those of the studies discussed previously.

Two issues must be kept in mind as generalizations are drawn from process-product studies. First, behaviors identified with effective instruction may not generalize across settings. For example, Evertson and others (1978) found that behaviors that were highly correlated with teacher effectiveness in mathematics were different from teacher behaviors that were associated with the effective teaching of English. No doubt this difference typically is less of a concern when considering teacher behaviors within a subject matter area than when one attempts to generalize across disciplines. Still, the importance of context variables other than subject matter has been illustrated in comparisons of the association between teaching behavior and student achievement in middle-class and working-class schools (Good and others, 1978).

A second caution to be applied when examining specific behaviors associated with effective teaching is that *cause and effect* conclusions should not be stated or implied. Such conclusions are important, but they must be determined by experimental studies.

## Experimental Work

A great number of experimental studies have focused on instructional methods in mathematics. Unfortunately, many of them have been isolated studies focusing on only one or two variables, using very small samples. Few have been based on previous process-product research that has comprehensively attempted to determine how more- and less-effective teachers vary in their behaviors.

An exception is the Missouri Mathematics Effectiveness Program (Good and Grouws, 1979a), an experimental study we conducted in fourth-grade classrooms, in which the treatment teachers taught using a system of instruction based on the results of a process-product paradigm and the research work of others. The system of instruction involved the following aspects:

1. Instructional activity was initiated and reviewed in the context of meaning.
2. A substantial portion of each lesson was devoted to content development (the focus was on the teacher actively developing ideas, conveying meaning, giving examples, and so on).
3. Students were prepared for each lesson stage to enhance involvement.
4. The principles of distributed and successful practices were used.

Pre- and post-testing with a standardized achievement test indicated that the performance of students in the experimental group was substantially better than performance of students in control classrooms. End-of-year achievement testing by the school district indicated that experimental classes continued to perform better than control classes three months after the post-testing on the mathematics subtests of a standardized achievement test. Also, experimental students had significantly better attiudes toward mathematics than did control students at the end of the treatment period, as measured on a ten-item attitude scale.

In a follow-up study in sixth-grade classrooms, we experimentally tested a revised and expanded instructional system (Good and Grouws, 1979b). In the fourth-grade study we observed that more learning gains were made in the knowledge and skill areas than in the problem-solving area. Thus the adjusted treatment was designed to improve verbal problem-solving performance without making excessive demands on teachers and without adversely affecting achievement gains in other areas. The additional teaching requests involved teachers daily giving attention to verbal problem solving by using several teaching techniques related to a given problem, such as estimating the answer and writing an open sentence. We also asked teachers to implement the regular instructional program (Good and others, 1977).

The results of the second experimental study showed that the problem-solving performance of students in the treatment group was significantly better than that of students in the control group. However, the achievement gains in other areas (knowledge and skill) were comparable. Although the raw gains of treatment students exceeded those of control students on the general achievement test, these differences were not statistically significant.

These data can be interpreted in two ways. First, it could be that implementing a general instructional program and using explicit strategies for improving verbal problem solving are too much for teachers to do in too short a period of time. An alternative explanation is that control teachers were using many of the strategies called for in the general instructional program. In the previous year teachers in the school district had been exposed to the general instructional program; thus, the information in the general instructional program was not unique, as was the emphasis on verbal problem solving (Grouws and Good, 1978).

Still, in separate field experiments it was possible to affect students' knowledge of mathematics, mathematical skills, and problem-solving abilities. What is less clear is whether or not these three aspects of mathematics instruction can be improved simultaneously or whether better training models would call for teachers to adapt their instruction progressively over time rather than attempting to make comprehensive changes at one point in time. Teachers who are asked to make several complex changes may find the accommodations so major that their instructional system is temporarily disrupted. Much more research is needed along these lines.

We have recently experimented with the instructional model in junior high school settings. Junior high teachers helped modify the program that we used in previous research in elementary schools. Essentially, the instructional program tested in junior high classrooms was very comparable to the one used in the elementary school research. Preliminary analyses suggest that the treatment classrooms out-performed control classrooms, especially in the area of performance on verbal problem solving items. Hence, it seems possible to intervene successfully in both elementary and secondary mathematics classrooms.

Importantly, teachers' reactions to the program have been very positive. This attitude has been reflected in anonymous data that we have collected (Good and Grouws, 1979b) and data that have been produced elsewhere (Keziah, 1980). Apparently, teachers find the requests presented understandable and sufficiently plausible that they are willing to try the program (our implementation data indicate that most teachers who participate in the experiment do use the program as indicated by their classroom behavior), and are willing to continue using the program after the experimental study has been terminated.

## Teacher and Student Effects

Peterson (1979) advocates the examination of instructional systems to determine which teaching behaviors best foster the achievement of particular types of students. It also seems reasonable to raise questions about the desirability of a given instructional program for use in changing the particular attitudes and skills of individual teachers. The instructional treatment program that we have been examining in the Missouri Mathematics Effectiveness Project can be described as focusing heavily on *active teaching*. When the effects of the program have been examined in terms of particular student types and particular teacher types (Ebmeier and Good, 1979), it is clear that certain students and certain teachers tend to do better using the treatment than do other combinations of students and teachers. Interestingly, the effects of the program on some of these combinations of student and teacher characteristics have been replicated by Janicki and Peterson (in press). It also seems that the classroom organizational structure (for instance, open-space plans vs. self-contained) also interacts with the treatment program (see, for example, Ebemeier and others, 1980). Without going into a detailed discussion of how student and teacher characteristics interact with the program, it should be noted that in our research context, all experimental groups have done better than all related control groups. However, the magnitude and importance of the differences are more evident for some teacher and student combinations than for others.

It should be evident that there is no single system for presenting mathematics concepts effectively. For example, some of the control teachers in our studies have obtained high levels of student achievement using instructional systems that differ from those presented in the program we have developed. Thus, there are many ways to effectively present mathematics.

However, the instructional program we have developed does seem to be a viable system that teachers are willing to implement. Also, it would seem to be an interesting alternative especially for those teachers who teach in self-contained classrooms and who enjoy an organized approach to instruction. One of the reasons that the program appears to be readily implementable probably is that the teaching strategies were derived from ongoing instructional programs. That is, the program was based upon what relatively effective teachers were already doing in the classroom. Hence, the program appears to have ecological validity and does not demand excessive amounts of teacher time and energy.

## Directions For Future Research

Our research approach is only one methodology for attempting to understand, describe, and improve mathematics instruction. One of the chief limitations to this method of studying mathematics is that one can only study teaching practice as it presently exists. Clearly, many exciting ideas for improving mathematics instruction are techniques that have yet to be implemented, and there is much room for creative theorizing about instructional strategies. However, as has been noted elsewhere (Good, 1980), there is considerable variation in teaching behavior in American classrooms. Indeed, one can view the abundant variation in instructional strategies as a rich source of naturally-occurring experiments. The process-product paradigm represents an important research methodology to the extent that teachers' instructional behavior and their effects on students vary in important ways.

At present, most of the process-product research has focused on an examination of teachers who consistently obtain more and less student achievement than do other teachers teaching comparable students. In general, teachers have been selected on the basis of their ability to affect student scores on standardized achievement tests. However, there are many problems with standardized achievement tests—they must be relevant to the instructional goals that teachers are actually pursuing in their classroom instruction. To the extent that this criterion is met, standardized achievement tests represent a reasonable proxy.

It would seem that further use of the process-product paradigm in the study of mathematics instruction should be accompanied by the use of outcome measures other than standardized achievement tests. For example, Confrey (1978) has argued that an important outcome of instruction in mathematics is the conceptual system that students derive from the study of mathematics. It would seem instructive to determine if some teachers consistently help students to develop a more adequate conceptualization of mathematics than do other teachers. Erlwanger (1975) argued that children develop a personal system of beliefs and emotions about mathematics that presumably controls their mathematical behavior in the future. It is important to study how teachers affect students' belief systems, and one potential way to explore this topic is through process-product research. That is, we could attempt to identify teachers who have a distinct impact upon students' beliefs about mathematics. Although the process-product paradigm has been used to explore teaching behavior in terms of its effects on student achievement, we see no reason why the model could not be used to profitably explore students' performance in other areas (problem solving) or other alternative outcomes of mathematics instruction.

We want to emphasize that a process-product approach to the study of mathematics is not the only method appropriate for studying teacher effects, for it has a number of limitations as well as advantages. Our purpose has been to identify some of the useful aspects of this approach and to call for its continued use, along with other methodological strategies, for exploring mathematics instruction. However, if the model is to continue producing positive contributions to theory and research, it would appear necessary to explore other dependent/outcome measures and to integrate the focus on teaching behavior with analyses of student behavior and perceptions (for example, clinical interview strategies). The study of teacher and student variables could be profitably combined with an active examination of the mathematics content being presented. It is likely that different types of instructional and learning strategies would be more or less effective for instruction in particular mathematical concepts or beliefs.

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# VI. The Sex Factor

Elizabeth Fennema

At the 1978 Annual Meeting of the National Council of Teachers of Mathematics, I concluded a presentation entitled "Sex-Related Differences in Mathematics Achievement: Where and Why" with the following remarks (Fennema, 1978):

> What, then, can be said that is known about sex-related differences in mathematics and factors related to such differences? Certainly when both females and males study the same amount of mathematics, differences in learning mathematics are minimal and perhaps decreasing. Many fewer females elect to study mathematics and therein lies the problem. Factors which appear to contribute to this non-election are females' lesser confidence in learning mathematics and belief that mathematics is a male domain. In addition, differential teacher treatment of males and females is important.
>
> There is nothing inherent which keeps females from learning mathematics at the same level as do males. Intervention programs can and must be designed and implemented within schools which will increase females' participation in mathematics. Such programs must include male students, female students, and their teachers. Only when such intervention programs become effective can true equity in mathematics education be accomplished.

In addition, I had discussed spatial visualization and concluded that it was the only cognitive variable which might be helpful in understanding sex-related differences in mathematics. Most of those statements I still believe. Others I am not so sure about.

## Sex-Related Differences in Mathematics

There is a great deal of new information about women and mathematics, plus increasing concern at the action level—the schools. In 1978, the National Institute of Education funded ten major research projects that investigated a number of factors related to the issue. The Women's Educational Equity Act has funded many product development projects, which can be used at all levels of education, to increase females' participation in mathematics. Information about the issue has appeared in a wide variety of publications ranging from the *American Educational Research Journal* to *Chronicle of Higher Education* to *Ms.* magazine. NCTM has had a major task force charged with making recommendations to its Board of Directors. Lead articles about the status of women and mathematics have been in the *Mathematics Teacher* and the *Arithmetic Teacher*. The *Journal for Research in Mathematics Education* has recently published three articles and the editor reports an increasing number of submissions about the topic. Inservice programs designed to increase teachers' awareness are being held nationwide. A major strand at the Fourth International Congress on Mathematics Education was about women and mathematics. The issue is one of the most widely talked about in the mathematics education community since the "New Math." In short, knowledge about the importance of mathematics to females and the inequitable education in mathematics that females have received is easily found. Also available are some intervention programs that have demonstrated effectiveness. What has been the result of this? Is the problem solved? Let's take a look.

The most important place to look to see if change is taking place is in schools themselves. Here exists a major problem. For a number of years, I have been convinced that we cannot talk about what is going on in high schools on a nationwide or statewide basis, or even on a systemwide basis. The analysis of all data I have ever collected, as well as other analyses that I have seen, has led me to the conclusion that sex-related differences in mathematics must be examined on a school-specific basis. Some schools have been remarkably successful in helping females learn mathematics and to feel good about themselves as learners of mathematics. Other schools have not. Some schools have more females than males enrolled in advanced mathematics classes. In many schools, the reverse is true.

Because of the discrepancies which exist among schools in enrollment patterns and also in feelings toward mathematics by females and males, it is somewhat hazardous to generalize. Nevertheless, an examination of the data that do exist is interesting.

## Enrollment Patterns

Armstrong (1979) reported a major study whose purpose was to determine the relative importance of selected factors affecting women's participation in mathematics. She collected data from a stratified, random sample of the entire United States. One factor on which she collected data was participation in mathematics classes. She asked twelfth-grade students to check the appropriate boxes if they had taken or were currently enrolled in courses with specific titles. She concluded that few differences exis[illegible] course-taking patterns for males and females in the twelfth grade. Her [illegible] are shown in Figure VI-1.

**Figure VI-1. Sex Differences in Participation in High School Mathematics***

**Fall 1978**

| | Percentages | | |
|---|---|---|---|
| | Female | Male | Differences |
| General Math | 88.77 | 90.12 | 1.35 |
| Accounting/Business Math | 40.21 | 32.62 | −7.59 |
| Consumer Math | 25.32 | 28.78 | 3.46 |
| Pre-Algebra | 64.98 | 65.08 | .10 |
| Algebra I | 78.49 | 81.06 | 2.57 |
| Geometry | 55.29 | 59.36 | 4.07 |
| Algebra II | 42.15 | 53.72 | 11.57 |
| Trigonometry | 27.14 | 30.96 | 3.82 |
| Probability/Statistics | 4.86 | 9.48 | 4.62 |
| Computer Programming | 13.28 | 18.16 | 4.88 |
| Pre-Calculus | 18.00 | 21.49 | 3.49 |
| Calculus | 7.23 | 8.20 | .97 |

* Armstrong, J., "Women and Mathematics: An Overview of Factors Affecting Women's Participation (Table 3)," paper presented at Research on Women and Education Conference, Washington, D.C., November 1979. (Available from Education Commission of the States, 1860 Lincoln Street, Denver, Colorado 80295.)

More females than males take accounting. In every other category, more males have taken or are taking the class. There appears to be no dramatic difference in course taking. The Second National Assessment of Educational Progress collected data in a manner similar to Armstrong (Fennema and Carpenter, in press). These data indicate approximately the same trend as does the Armstrong sample.

Is the same trend evident when we look at data from a state sample? Wyoming recently completed such a survey, in which mathematics preparation was classified on six levels. Level 6, the least prepared, means students have had only general mathematics. Level 1, the highest level, means students have studied algebra I and II; synthetic and analytic geometry; trigonometry; logarithmic functions (common and natural) and their graphs; mathematical induction; algebra of functions; basic operations on matrices; and limits, continuity, and differentiation of polynomial functions. Figure VI-2 shows the seniors who had attained each level in 1978. At only the two lowest levels did females and males have the same preparation.

California also reports that a greater percentage of boys than girls take four years of mathematics (24 percent male vs. 17 percent female), while Wisconsin reports a 6:4 ratio of male : female in their most advanced course (Perl, 1980). Overall, these enrollment data are moderately encouraging.

**Figure VI-2. Percent of College-Bound Seniors Attaining Each of Six Levels of Mathematical Preparedness***

| | Percent | |
|---|---|---|
| Level | Female | Male |
| 1** | 15 | 30 |
| 2 | 21 | 36 |
| 3 | 54 | 64 |
| 4 | 71 | 78 |
| 5 | 93 | 94 |
| 6 | 100 | 100 |

* Kansky, B., and Olson, M., *Mathematical Preparation Versus Career Aspirations: Study of Wyoming 1978 High School Seniors* (The Science and Mathematics Teaching Center, Box 3992 University Station, Laramie, Wyoming 82071), p. 13.

** Level 1 is the highest level of mathematical preparedness.

What happens when we look at individual schools rather than compiling across the nation or a state? In 1977, my colleagues and I began developing an intervention program designed to increase high school girls' enrollment in mathematics classes. This project was funded by the Women's Educational Equity Act Program and included a major evaluation component. In order to carry out this evaluation, we wanted to use the program in schools having an imbalance in enrollment by sex in advanced mathematics courses. I assured my colleagues that finding such schools would be no problem since all the literature (including some written by me) said such an imbalance almost always existed. Imagine my surprise (and embarassment) to find that in about one-third of the schools either an equivalent number of females and males or more females than males were enrolled in advanced math classes. I must add that in most of these schools there were not many advanced classes and there were few students of either sex enrolled in them. However, I am convinced now that while enrollment trends may be encouraging on a broad scale, it is only by looking at individual schools that meaningful assessment of females' enrollment in advanced math courses can be made.

## Mathematics Achievement

A more critical issue is what current data tell us about whether sex-related differences in achievement exist when the number of mathematics courses boys and girls have studied is held constant. In other words, do girls and boys who report that they are currently enrolled in or have been enrolled in the same mathematics courses achieve equally in mathematics? If these differences in achievement exist, are they large enough so that we should be concerned?

Studies which shed light on mathematics learning by females and males are becoming increasingly sophisticated in at least two ways: (1) mathematics course taking is being considered an important variable to control, and (2) the cognitive complexity of the items used to measure learning is being included. Prior to 1978 studies which considered both of these were basically not available. Now there are four such studies that deal with relatively current data. Two of them reflect information gathered from nationwide samples. The California State Assessment of Mathematics was done in 1978. Students in Grades 6 and 12 were tested on a variety of content areas with items of differing cognitive levels. Comparisons in achievement level were made among groups who reported studying the same number of mathematics courses. A committee was named to evaluate the results and concluded the following about sex-related differences:

> An analysis of the results by sex showed that girls do consistently better than boys in computations with whole numbers, fractions, and decimals. The girls also outperformed boys in simple one-step word problems. However, the committee found that boys typically scored higher on word problems that were either multiple-step problems or required more reasoning ability.
>
> In geometry, the girls scored higher than boys on questions involving recall and identification of geometric shapes, while boys achieved higher than girls on items dealing with spatial relationships and reasoning ability. In measurement, the girls generally scored higher than boys on problems dealing with money; however, boys generally performed better than girls on the other questions.
>
> At the twelfth grade, the relative performance of boys and girls was compared taking into consideration the amount of mathematical preparation of the particular courses that students had completed. The committee noted that the girls continue to outperform boys at the twelfth grade in whole number and decimal computations. However, the girls do not keep up their relative achievement level in fraction computation at the twelfth grade. The committee observed that girls were considerably lower in the

skill areas of measurement, geometry applications, and probability and statistics.

Females tended to achieve higher than males on lower level cognitive tasks, while males scored higher on more complex cognitive tasks.

Armstrong also investigated achievement differences. She compared females and males who had taken the same math courses and concluded: "Twelfth grade males scored significantly higher than females on the problem solving subtest. Thirteen-year-old females scored significantly higher on the lower level mathematical skill of computation." The mathematics Assessment of the Second National Assessment of Educational Progress indicated also that females were somewhat better in computational tasks than were males. Males out-achieved females in higher level cognitive tasks (Fennema and Carpenter, in press).

These four major studies have made me somewhat uncomfortable with the idea that the only thing we have to do is to ensure that females continue to enroll in mathematics during high school. I am still convinced that the majority of females will achieve at the same level as the majority of males if they elect to study the same amount of mathematics. However, differences in achievement on high level cognitive tasks deserve more direct investigation both with large samples and at the individual school level.

## Related Variables

In 1978 I reported that females felt less confident in learning mathematics than did males, and they tended to believe that mathematics was less useful to them than to males. These are still two major variables that explain sex-related differences. My colleagues and I are currently involved in a longitudinal study attempting to identify influences on the development of feelings of confidence. Other than to re-emphasize its importance, I have nothing new to report. Belief in the usefulness of mathematics can be changed and later I will talk about how and why.

One major variable that might help explain females' falling behind in learning higher level skills has to do with the practice of such skills. Enrollment in both mathematics-related courses (such as computer science, probability and statistics) and in science courses which use a great deal of mathematics such as chemistry and physics may contribute to the difference. The female : male ratio in such classes is much higher than in traditional mathematics classes. In these related classes, mathematics is used or applied at the same cognitive level at which males are out-achieving females. We learn what we practice: if males tend to participate in higher level skills in other classes, then they will undoubtedly learn them better.

It has long been a hypothesis of mine that teachers expect males to be better problem-solvers than are females and that boys, more than girls, are encouraged to engage in problem-solving activities. If this hypothesis is true, girls would tend to engage in lower level cognitive activities more than do boys. A result of such differential practice would be the different achievement by cognitive level that has been observed.

We do know that teachers are the most important educational influence on students' learning of mathematics. From entry to graduation from school, learners spend thousands of hours in direct contact with teachers. While other educational agents may have influence on educational decisions, it is the day-by-day contact with teachers that is the main influence of the formal educational institution. Part of the teachers' influence is the learners' development of sex role standards. These sex role standards include definitions of acceptable achievement in the various subjects. The differential standards for mathematics achievement is communicated to boys and girls through differential treatment as well as differential expectations of success.

Many studies have indicated teachers treat female and male students differently. In general, males appear to be more salient in the teachers' frame of reference. Teachers interact with males more than with females in both blame and praise contacts (Becker, 1979). More questions are asked of males by teachers. Males are given the opportunity to respond to more high level cognitive questions than are females (Fennema and others, 1980a).

High achieving girls seem particularly vulnerable to teachers' influence. One major study (Good and others, 1973) indicated that high achieving girls received significantly less attention in mathematics classes than high achieving boys. On the other hand, many girls who have been accelerated in mathematics report positive teacher influence (Casserly, 1980) as a cause of their success. This influence was manifest by teachers being 'sex-blind' in the treatment of girls. Teachers treated males and females just alike and had high expectations for females, as well as males.

Another theory that might help in understanding the sex-related differences in mathematics is attribution theory, although I must urge caution in accepting this theory in a simplistic way. Attribution theory has to do with the perceived causes of success and failure experiences. The model that appears to be the most useful to educators as an aid to understanding achievement-related behavior is the one proposed by Weiner (1974). In this model, attributions of success and failure are categorized into the matrix shown in Figure VI-3, with locus of control being one dimension and stability the other.

**Figure VI-3. Locus of Causation**

| | | INTERNAL | EXTERNAL |
|---|---|---|---|
| STABILITY | STABLE | Ability | Task |
| | UNSTABLE | Effort | Luck |

Attributions of past successes and failures to certain of the Weiner categories have been shown to be related both to task persistence and task choice (Bar-Tal, 1978). If one attributes success to internal causes, particularly the internal stable cause of ability, then one can expect success in the future and will be more apt to approach or persist at certain tasks. If, however, success is attributed to an external cause, success in the future is not assured and one will avoid the task. A somewhat different situation is true of failure attributions. If one attributes failure to unstable causes, failure can be avoided in the future so the tendency to approach or persist at tasks will be encouraged. Attribution of failure to a stable cause, on the other hand, will lead one to believe that failure can't be avoided.

Although we should be extremely careful of overgeneralizing data and concluding that all males behave one way and all females another way, many studies have reported that females and males tend to exhibit different attributional patterns (Deaux, 1976; Bar-Tal and Frieze, 1977). In a somewhat simplistic summary, males tend to attribute successes to internal causes and failure to external or unstable causes. Females tend to attribute successes to external or unstable causes and failure to internal causes.

These attributional patterns have also been linked to a pattern of behavior called "learned helplessness"—the condition in which failure is viewed as inevitable and insurmountable. This condition results in lowered motivation to persist. Females are more likely than males to display learned helplessness (Dweck and others, 1978).

The principles of attribution theory and learned helplessness can be applied to the problem of math avoidance—the lack of persistence in mathematics related activities. It appears reasonable to hypothesize that if a student attributes successful performance in mathematics to ability, the likelihood of persisting in mathematics is higher than if that success were attributed to an unstable cause such as effort or luck. Conversely, when failure is attributed to ability, lowered persistence will result. The differential in male/female enrollment in mathematics—the lack of persistence among females—might be partially explained by the fact that females, more than males, attribute successful performance to unstable causes and unsuccessful performance to stable ones. A recent study by Wolleat and others (1980) indicated that this is true. Females, when compared to males, exhibited more of the learned helplessness pattern in their attribution of success and failure in mathematics. They were more likely than males to use effort (unstable) and less likely than males to use ability (stable) to explain their successes. When explaining mathematics failures, females invoked the attributions of ability and task difficulty (both stable) more strongly than did males.

Another variable which many believe might help explain both differential course taking and differences in achievement is spatial visualization. Spatial visualization involves visual imagery of objects, movement of the objects or changes in their properties. In other words, objects or their properties must be manipulated in one's 'mind's eye,' or mentally. The relationship between mathematics and spatial visualization is logically evident. Starting at about adolescence, male superiority on tasks involving spatial visualization is found. Many are finding that spatial visualization is related to mathematics achievement differently for males and females (Sherman, 1979).

Currently, my colleagues and I are engaged in gathering data about how mathematics learning is dependent upon spatial visualization. It appears evident that tasks which measure spatial visualization skills have components that can be mathematically analyzed or described. From such an examination, we could hypothesize a direct relationship between mathematics and spatial visualization. An item from the space relations portion of the Differential Aptitude Test (Bennett and others, 1973) requires that a 2-dimensional figure be folded mentally into a 3-dimensional figure. Another spatial visualization test requires that 2- or 3-dimensional rigid figures be rotated and translated to specified locations. The activities required by those tests can be described as mathematical operations. Yet this set of operations is only a minute subset of mathematical ideas which must be learned and, indeed, one could go a long way in the study of mathematics without these specific ideas.

The hypothesis that my colleagues and I are currently investigating is that the critical relationship between mathematics and spatial visualization is not direct, but quite indirect. It involves the translation of words and/or mathematical symbols into a form where spatial visualization skills can be used. For instance, consider the following problem:

> A pole 12 feet long has been erected near the bank of a lake. Two and a half feet of the pole have been hammered down into the bottom of the lake; one half foot is above the surface of the water. How deep is the lake?

For children of 11 and 12 years of age, this is a moderately difficult problem. You must add the lengths of two pieces of the post and then subtract that length from the total length, that is, $2\frac{1}{2} + \frac{1}{2} = 3$ and $12 - 3 = 9$. Keeping track of the steps and sequencing them accurately is not easy. Consider the problem from a spatial visualization perspective. If you can visualize in your mind what is involved, the solution of the problem then becomes simpler. An image would enable a person to move the pieces above and below the water together. Then that length could be subtracted from the total length in order to get the correct answer.

Consider a symbolic problem met by children of the same age: $\frac{1}{2} + \frac{1}{3}$. While it can be solved totally with symbols, children of this age, because of their developmental level, often have trouble really understanding the symbolic process involved. If it could be visualized in the mind, spatial visualization skills could be used and the answer found more easily.

We know that females tend to score lower than males on spatial visualization tests. What we do not know is whether females differ from males in their ability to visualize mathematics, that is, in the translation of mathematical ideas and problems into pictures. Neither do we know if good spatial visualizers are better at these translations than are poor spatial visualizers. However, I am increasingly convinced that there is no direct causal relationship between spatial visualization skills and the learning of mathematics in a broad general sense. While I am continuing to investigate the impact of spatial visualization skills, I am less convinced than I once was that spatial visualization is important in helping understand sex-related differences in the studying and learning of mathematics.

In American schools, classrooms don't appear to use mathematical representations which either encourage or require the use of spatial visualization skills. While some primary mathematics programs encourage the use of concrete and pictorial representations of mathematical ideas, by the time children are 10 or 11 years old, symbolic representations are used almost exclusively. Perhaps boys, more than girls, use the concrete representations during primary years and, thus, develop higher skills in using spatial visualization in learning mathematics. As far as I know, however, no one is investigating such a hypothesis.

## Interventions

Can schools be changed so that females elect to study more mathematics and learn mathematics as well as do males? All too often, comments are addressed to me that imply that schools alone can't do much. The argument goes like this. Because the studying of mathematics is stereotyped male, and because stereotyping of sex roles is so deeply embedded in society, schools are powerless to improve females' studying of mathematics until society changes. Let me say as emphatically as I can that that argument is fallacious. Schools can increase females' studying of mathematics. Let me cite some evidence that shows strongly that schools can be effective.

Two intervention programs in particular have been intensively evaluated. The first program is called *Multiplying Options and Subtracting Bias*. The rationale used in its development is that merely telling high school females about the importance of mathematics is insufficient. Forces that influence these girls to make their decisions are complex and deeply embedded in societal beliefs about the roles of males and females. Asking females to change their behavior without changing the forces operating on them would place a very heavy burden on their shoulders. What should be done is to change the educational environment of these females so that they are enabled to continue their study of mathematics beyond minimal requirements. This environment is composed of several significant groups of people: mathematics teachers, counselors, parents, male students, and the female students themselves. *Multiplying Options and Subtracting Bias* was designed to change these significant groups' beliefs about women and mathematics as well as to change each group's behavior.

*Multiplying Options and Subtracting Bias* is composed of four workshops: one each for students, teachers, counselors, and parents. Each workshop is built around a unique version of a videotape designed explicitly for the target audience. Narrated by Marlo Thomas, the tapes use a variety of formats, candid interviews, dramatic vignettes, and expert testimony to describe the problem of mathematics avoidance and some possible solutions. The videotapes and accompanying workshop activities make the target audience aware of the stereotyping of mathematics as a male domain which currently exists, females' feelings of confidence toward mathematics, the usefulness of mathematics for all people, and differential treatment of females as learners of mathematics. Discussed specifically are plans for action by each group. These two-hour workshops are designed to have an impact on a total school.

The program has been evaluated extensively and its use has significantly increased females' enrollment in mathematics courses (Fennema and others, 1980a). Exposure of *Multiplying Options and Subtracting Bias* can substantially influence students' attitudes about mathematics, the stereotyping of mathematics, and students' willingness to take more mathematics courses.

The other intervention program was developed, planned, and implemented by the San Francisco Bay Area Network for Women in Science (now called the Math/Science Network). The Network is a unique cooperative effort of scientists, mathematicians, technicians, and educators from 30 colleges and universities, 15 school districts, and a number of corporations, government agencies, and foundations. The goal of the Network is to increase young women's participation in mathematical studies and to motivate them to enter careers in science and technology.

Seven conferences developed by this network were held in the spring of 1977 and 1978 to increase the entry of women into mathematics- and science-oriented careers. These one-day conferences consisted of a general session with a panel or main speaker, one or two science/math workshops, and one or more career workshops that provided junior and senior high school girls opportunities to interact with women working in math/science-related fields.

The conferences were evaluated in a study involving 2,215 females who had volunteered to attend. Pre- and post-conference questionnaires were administered and responses analyzed. The evaluators concluded that "the conferences (1) increased participants' exposure to women in a variety of technical and scientific fields, (2) increased participants' awareness of the importance of taking mathematics- and science-related courses, and (3) increased participants' plans to take more than two years of high school mathematics" (Cronkite and Perl, 1979).

Evaluations of these intervention programs indicate quite clearly that it is possible to change females' mathematics behavior, and to do so in relatively short periods of time.

Some schools are remarkably more effective than others in persuading females to attempt high achievement in mathematics. Casserly (1980) identified 13 high schools which had an unusually high percentage of females in advanced placement mathematics and science classes. She concluded that the schools had identified these girls as early as fourth grade and the school teachers and peers were supportive of high achievement by the females.

## Questions

I would like to conclude with the following questions and answers. Perhaps your answers would be different, but I challenge you to at least think about mine.

**Q:** Is the sex factor a reality in mathematics education?

**A:** Yes. Females are receiving an inequitable education in mathematics in many schools. Not only do they elect to study mathematics less than males do, there is some evidence that in a very important part of mathematics learning, they learn less than do males.

**Q:** Can schools do anything about improving the mathematics education of females?

**A:** Yes, a great deal. Each school must first find out what its situation is with respect to enrollment and achievement and then plan interventions which specifically address that unique situation.

**Q:** Can individual teachers do anything?

**A:** Yes. By becoming truly sex-blind in expectations—by increasing their awareness of all students as individuals who have unique needs which must be met in order to help students achieve at their highest level.

**Q:** What are the implications of research on sex-related differences for the curriculum of the 80s?

**A:** Sex-related differences in mathematics can, and should, be eliminated. Equity in mathematics education for females and males is an achievable goal. In order to achieve this goal, each and every school must consider its own specific situation. Much help is available, but the motivation and direction must come from within each school.

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# Response

Grace M. Burton

At NCTM's 1978 annual meeting, the strand "Women and Mathematics" attracted a large cross section of the mathematics education community. Topics ranged from mathematics anxiety to problem-solving ability, from sexism in mathematics textbooks to the distribution of females in leadership roles. Like Fennema, I concluded my presentation with a call for commitment:

The inability of women to succeed

> at mathematical endeavors is a bit of hallowed mythology in our folklore, but there are those who seek to change that. Join us. Encourage each mathematically talented Susie you know to excel in mathematical endeavors and to be *proud* of her ability.

I believe we are presently seeing the fruition of that interest in sex-related differences which had resurfaced in the mid-70s. We are beginning to have the hard data to support—or refute—some of our hunches. Our openness to the findings of research may be tested as some of what we "know" turns out to be not so at all. It is important that we both evaluate the new findings and re-evaluate our beliefs in the light of those findings deemed valid.

One difficulty in evaluating research in the area of sex-related differences is the sheer quantity of the material currently appearing in professional journals. Relevant information is scattered across many disciplines including economics, sociology, neurology, developmental psychology, anthropology, linguistics, philosophy, and biology. Definitions vary from one study to another, even in such basic terms as "feminine," "problem solving," and "spatial ability." Nonsignificant differences tend to be under-reported while differences tend to be over-reported. Much of the data have been collected by self-report. Birth order, race, and socioeconomic class, all of which may be significant factors, are confounded with sex. Population equivalence cannot be relied upon. Drawing conclusions from older studies presents other difficulties. Extrapolations from animal research, generalizations from the observations of one sex to both sexes, misinterpretations in secondary sources, and societal changes—all have contributed to the present state of the art. At base, though, is the point raised by Fennema—all we can learn from these studies are generalities about males *as a group* or females *as a group*. Traditional research can tell us nothing about what a particular male or female can or will choose to do or be.

It may be that the very vigor of this research activity has given rise to what Fennema calls the new mythology: that is, males and females are basically different in cognitive and psychological make-up. There is now a real danger that this mythology will play a part in educational decision making. Luckily, at least in those institutions receiving federal financial assistance, schools are prohibited from offering single-sex courses or extracurricular activities. Were this not the case, we might see a rash of "Trig for Girls" or "Calculus for Girls" classes. Such efforts to accommodate group differences are shortsighted and inappropriate at best. More often, they are both ineffective and deprecating. As we look back with humor on the 19th century assertions that women actually had significantly different breathing apparatus or nervous systems, we must be careful that we do not subscribe to more modern but possibly equally ludicrous beliefs.

### Enrollment Patterns

The study of mathematics is vital to the intellectual development and career progress of both male and female students. I would be among the first to deny that all female students are less likely than all male students to elect to study mathematics when it is no longer a required subject. In some schools, the number of each sex in advanced courses is about equal. I would suspect, although I have no data to back me up, that in those schools where large numbers of young women are continuing to study mathematics beyond the required courses, someone (or more than one someone) is consciously doing something to help this change along.

I firmly believe in the power of the individual to effect change. Each individual has the opportunity to alter the status quo in the direction of greater good. And of course I believe that encouraging each student to actualize his or her intellectual potential to the fullest possible degree is a "good." I repeat without apology, "The place to improve the world is first in one's own heart and head and hands and then to work outward from there" (Pirsig, 1975). Each of us have the obligation to do just that. To accept "More males than females take upper math courses" as an unchallengeable assumption is to abdicate that responsibility. Those schools in which female enrollment patterns are different from the norm should be studied, the contributing factors identified and promulgated, and, where possible, modeled in other schools.

### Mathematics Achievement

It is hard to accept the result that females tend to achieve higher than males on lower cognitive skills, yet lower than males on higher cognitive skills. Here again, however, we must be cautious about translating that finding into "Girls can't do as well as boys on complex cognitive tasks." I concur with Fennema's statement that the probable causes underlying the reported differences require direct investigation. We must not lose sight of the individual student who has the potential to perform differently from what group norms would lead us to expect. Regardless of the social convention, there have always been women who have delighted in the study of mathematics and who have achieved success despite the fact that cognitive activity was not considered the province of "the gentle sex." Femaleness must not be taken as presumptive evidence of inability to achieve at high mathematical levels.

### Related Variables

The strength of variables related to mathematics course-taking such as confidence and belief in the usefulness of mathematics is evident both from research and less structured observation. These factors each school can and should address. Attribution patterns are a newer focus. Building on this current research, those teachers who have long believed in the power of an "Of course you can!" philosophy for themselves and their students may be able to refine their thinking and apply it more effectively.

### Interventions

In the concern for well-designed studies and carefully-written reports, it is easy to underestimate the role of the educational practitioner who has the power to speed or retard change in the educational system. If only teachers could be convinced of their power for good! They are *the* powerful influence in the lives of their students for the successful (or unsuccessful) achievement of academic goals, for the development of positive (or negative) attitudes, and for the embracing of (or the escape from) further exploration of mathematics.

Teachers become even more powerful when they know the result of research.

Teachers familiar with research on "wait time" (Rowe, 1978) are unlikely to transmit an "I know your kind can't do it" message by not allowing time for students to answer questions. Teachers who know the Good, Sikes, and Brophy (1973) findings are more likely to monitor their attending behavior. Teachers who have learned of attribution research will perhaps recognize the subtle differences in those students who blame study patterns for their failure and those who say, "I don't have a math mind." Teachers who have studied the effect of spatial visualization may take special pains to incorporate appealing activities in this dimension in their classes.

I certainly agree that spatial ability is not crucial to the pursuit of mathematics. Now that their diplomas and/or careers are relatively secure, several students and colleagues have told me they never *could* see those rotating shapes in calculus, and that they analyzed their way through projective geometry. On the other hand, certain tasks are facilitated by the ability to mentally rotate or translate figures. If an individual's spatial ability can be developed, develop it we should —not because it will eliminate a sex difference but because it will expand that individual's problem-solving repertoire and enrich his or her life.

Effective intervention programs on all educational levels are needed, and it is heartening that they are being developed, tested, and disseminated. Each and every teacher of mathematics within a school must make a concentrated effort to help each and every student make whatever cognitive and affective strides he or she can. Enlisting the support of counselors, teachers and parents as well as both male and female students is a most promising direction for such programs to take. It recognizes the basic fact that each of us is part of a system, and change in one part of the system effects all the rest of it. It is only when many individuals each make a conscious decision for positive change that such change will occur. Those many individuals must be informed of the existence, extent, and impact of the traditional mythology that women neither can nor should do mathematics.

There are many successful strategies (Menard, 1979). Some appropriate to the local conditions should be chosen, modified to meet the needs of the individual schools, and implemented. Gaining support at the administrative level for these new directions will facilitate the achievement of the desired goals, if only because administrators have the resources to help things happen once they are committed to the value of an idea. In this case, that valuable basic idea is fairness.

Of course, what I am suggesting involves seeing each student as a person first and accepting him or her without comfortable prejudgment and ready-made expectation due to sex, race, social class, sibling performance, or any other factor. It means throwing off any preconceived notion that individuals in any group are by nature logical or illogical, excited or bored by mathematics, ambitious or passive with respect to career. It means remembering there is a vast amount of variation in any group, and that it is an intellectually indefensible act to ascribe characteristics to an individual solely on the basis of group membership.

If each of us in our own spheres of influence demonstrate firm commitment to the importance of the individual, and translate our knowledge from research into action, there may be no need in the future to consider the question, "Is there a sex factor in mathematics education?" We will have provided the best learning environment possible for each student—regardless of sex.

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# VII. Problem Solving

Mary Grace Kantowski

The first of NCTM's *Recommendations for School Mathematics of the 1980s* (1980) states, "Problem solving must be the focus of school mathematics in the 1980s." This recommendation not only indicates the importance of problem solving, it also implies that a concerted effort is needed in order to establish problem solving as an integral part of the mathematics curriculum. Before we look ahead to what the 80s hold for problem solving, let us look back to where we were at the beginning of the last decade to see what we have learned about problem solving and where we stand today.

In a comprehensive review of problem-solving research written just prior to the beginning of the 70s, Kilpatrick (1969) noted that "Since the solution of a problem—a mathematics problem in particular—is typically a poor index of the processes used to arrive at the solution, problem-solving processes must be studied by getting subjects to generate observable sequences of behavior." He noted that psychologists had devised numerous techniques for studying problem solving, but that mathematical problems were seldom used in such research. Furthermore, he noted that the larger question of how subjects adapt various heuristic methods to different kinds of problems remains virtually unexplored. The situation described by Kilpatrick a decade ago no longer exists. Mathematics educators *have* been studying problem solving using mathematical problems, and a great deal of effort has been expended in the research community in the study of problem-solving processes.

Until the Kilpatrick report, much of the research and development in problem solving focused on the actual *solution* to the problems or the *answers to the exercises*. Researchers looked at how many problems were solved correctly, without regard for how the solution was attained or how close a student may have come to a correct solution. More recently, research has begun to examine *processes* or *the set of steps students use to find a solution*. In this form of research, the protocols (everything a student says or does as he or she solves a problem) are collected during individual interviews. Although difficult and time consuming, this emphasis on studying how solutions are arrived at has uncovered some interesting regularities common to correct solutions. It has been found, for example, that correct solutions to problems involve setting up a plan, however brief, for the solution (Kantowski, 1977, 1980). Another finding is that different students approach the same problem in a variety of ways, indicating the existence of a style or preference. This would suggest that curriculum developers and textbook authors should consider instruction in problem solving that includes a variety of approaches.

Another change since 1970 has been in the expansion of the meaning of problem solving. At the beginning of the last decade, problem solving to most people meant the solving of verbal or word problems. Although verbal problems remains an area of great interest in the mathematics education community, the term problem solving now includes other problem types such as *nonroutine mathematics problems* and *real (application) problems*.

To many classroom teachers and other educators a *problem* is simply a word problem or an exercise stated in verbal form. Word problems found at the ends of chapters in mathematics books fall into this category. An example of such a verbal problem might be the following:

> Maria bought a hamburger for $.90 and a coke for $.30. If the local sales tax is 5%, how much change should she receive if she gives the clerk $2.00?

Such problems are easily solved by application of algorithms that are a part of standard instruction.

To other educators a problem exists if a situation is nonroutine, that is, if the person attempting the problem has no algorithm at hand that will guarantee a solution. He or she must put together the available knowledge in a new way to find a solution to the problem. Such problems are subjective, that is, what is a nonroutine problem for one person is actually an exercise or a routine problem for another. For most middle school students, the following would be a nonroutine problem:

> Maria has exactly $3.00 and would like to spend it all on her lunch. The menu includes hamburgers at $.90, hot dogs at $.80, onion rings at $.60, french fries @ $.50, and colas @ $.30, $.40, or $.50. The sales tax is 5%. What could Maria have for lunch?

In solving this problem a student has no simple calculation algorithm to follow. The possibilities must be tabulated and some trial and error attempted. Moreover, more than one solution is possible.

In general, a nonroutine problem may be defined as a question which cannot be answered or a situation that cannot be resolved with the knowledge immediately available to a problem solver. In effect, a problem is a situation which differs from an exercise in that the problem solver does not have a procedure or algorithm which will certainly lead to a solution (Kantowski, 1974). That is not to say that such an algorithm does not exist, simply that it is not known to the problem solver at a given point in time. In fact, the solution to a problem may provide a problem solver with algorithms for future exercises.

A third type of problem can be called *applications* or "real problems." Projects such as Unified Science and Mathematics for the Elementary Schools (USMES) deal with real problems, and several curriculum projects, notably Usiskin's *Algebra Through Applications,* have emphasized applications.

## Verbal Problems

A comprehensive review of research related to verbal problem solving was undertaken by Sowder and others (1978). Most children in the elementary school are introduced to problem solving through verbal problems. After having been introduced to algorithms in some content area, the next logical step in instruction is to introduce the student to a problem in which the algorithm is being used or applied, to observe if a student is able to use the algorithm correctly, and more importantly, whether he or she is able to select the correct algorithm to use.

If we look at the first verbal problem stated above we see that in this case a student needs to select from and apply several algorithms:

| | |
|---|---|
| Maria bought a hamburger *and* cola: | |
| Purchases must be totaled | .90 |
| (Addition must be selected and applied) | .30 |
| | $1.20 |
| Sales tax percent must be changed to a decimal | .05 |
| Amount of sales tax must be found | $ .06 |
| (decimal multiplication must be selected and applied) | |
| Sales tax must be added to price | $1.20 |
| (addition must be selected and applied) | .06 |
| | $1.26 |

| | |
|---|---|
| Finally, the total cost must be subtracted from amount given clerk | $2.00<br>1.26 |
| (subtraction must be selected and applied) | $ .74 |

This kind of problem-solving activity is at a much higher cognitive level than simply performing a computation. There are three steps in the problem-solving sequence for each part of the problem. The student must recognize the structure of the verbal problem, select an appropriate algorithm, then correctly apply the algorithm. Studies give clear evidence that all three steps are necessary for successful problem solving in some measure and that instruction must place emphasis on each of these steps.

Skill in computational processes is necessary for solving problems (Knifong and Holton, 1976; Meyer, 1978). However, having these skills does not guarantee successful problem solving. Results from the second assessment of the National Assessment of Educational Progress support this idea. Although 76 percent of the 9-year-olds and 96 percent of the 13-year-olds could subtract a two-digit number from a two-digit number, only 59 and 87 percent, respectively, could solve a simple application problem using the same subtraction exercise. The results were even more dramatic when the operation was multiplication of fractions. In the case of the 13-year-olds, the percentage of correct responses dropped from 69 on the computation exercise to 20 on the related application problem. Clearly, a factor other than computational skill is involved in solving verbal problems. Results of studies such as those cited point to a need for instructional methods that emphasize something in addition to computation.

## Nonroutine Mathematics Problems

In general, nonroutine problems are problems for which a problem solver knows no clear path to the solution and has no algorithm which can be directly applied to guarantee a solution. In the case of the problem stated above, it is not clear from the statement of the problem what the selection of lunch items should be. The student must either use some trial and error to put items together to sum to a given total or organize the data into a table of possible combinations that would give the desired result. Such problems are at a higher cognitive level than simply selection and application of algorithm.

Nonroutine problems are important for several reasons. Experience in solving nonroutine problems can help students transfer methods of problem solving to new situations. Such experiences can also help students grasp the meaning of mathematical structure and develop the ability to see the mathematics in a given situation. Several recent studies give us a good deal of information on the status of solving nonroutine problems. First, it has been found at the elementary, secondary, and postsecondary levels that without specific instruction in techniques for solving nonroutine problems, most students do not know how to approach such problems and do not appear to use strategies in their solution (Lester, 1975; Kantowski, 1974, 1980; Schoenfeld, 1979). In a finding similar to that of Meyer (1978) cited above, Webb (1979) found that conceptual knowledge and heuristic strategy components, among other factors, interact in successful problem solving. This means that it is not simply computational skill and the knowledge of how to apply algorithms that are important in problem solving; it is also important for a student to be able to plan effectively and to use other heuristics such as organizing data into tables and drawing effective diagrams.

Several processes appear to be important in solving nonroutine problems. The solution set-up is the most difficult of the problem solving stages and the most crucial part of the solution (Kulm and Days, 1979). Solution set-up refers to a variety of manipulations of data that could lead to a solution such as organizing data into a table, grouping data into similar sets, or formulating an algebraic equation which would be useful in solving the problem. The ability to set up the problem is related to its successful solution.

Another process, which is closely related to solution set-up and has been emphasized in research dealing with nonroutine problems, is that of planning. In deciding on a plan for solution, a problem solver tries to find a relation to other problems solved previously and decides on a method of solution to try to follow. Plans often precede the solution set-up. Although planning does not always ensure a correct solution, psychologists (for instance, Greeno, Anderson, Rissland), as well as mathematics educators (Webb, 1979; Kantowski, 1977) have found that planning is related to successful problem solving and that most successful solutions of nonroutine problems show some evidence of planning.

Another variable studied in research dealing with the solution of nonroutine problems is that of transfer—memory for and application of methods used in previously solved related problems. Kulm and Days (1979) found that a general-specific sequence of problem presentation—that is, a sequence of problems in which a more general problem is presented initially and followed by a specific case and a similar problem—produced significant transfer of information and that content played a significant role. Both equivalent and similar problems resulted in transfer, although for different type problems. Kantowski, too, in a study of nonroutine problems (1980), found that some reference to related problems existed in a significant number of correct solutions. For example, in one of the instructional sessions, the solution to a problem used in instruction generated the pattern of triangular numbers (1, 3, 6, 10 . . .). Later in the study, many students referred to the pattern of triangular numbers that they had seen generated as they tried to solve problems that looked unrelated on the surface. Some students remembered the pattern; others remembered how the pattern was generated (by adding successive natural numbers to the previous element of the set).

## Real Problems

A real problem involves a complex real-life situation that must somehow be resolved. Often there is not an exact solution, but one that is determined to be optimal to fit the conditions. Real problems include traffic flow problems, and problems dealing with financing school functions and effectively utilizing available space. In arriving at a solution to a real problem, students often solve a variety of brief application problems that include substantial computation.

Perhaps the best known study of real problem solving is that done by the evaluators of the USMES program (Shann, Unified Science and Mathematics for the Elementary School, 1976). In this program, students in a class work together in small groups in an effort to solve a "real problem" such as the design of a soft drink that could be served at a party or event that would satisfy most of the group. In solving such problems, students become involved in collecting and compiling data in making decisions about what and how much to purchase and what to charge if the soft drink is to be sold. Computational skills are used in application situations so that the use of the skill becomes meaningful to the student. Among the significant results of the program were the wider repertoire of successful problem-solving behaviors exhibited; larger amounts of time spent in more active, self-directed, and creative behavior; overall higher (although not statistically significant) means on basic skill tests; and significant positive attitudes toward mathematics.

## Research on Instruction and Development

### Ideas for Instruction

Problem-solving ability develops slowly over a long period (Wilson, 1967; Kantowski, 1974) and grows with experience in solving problems. Therefore, for most students, except perhaps the most gifted in mathematics, systematically planned instruction is an essential factor in the development of problem-solving ability.

A variety of factors seem to affect the ability to be a successful problem solver. In all types of problems with which research has been concerned, the three variables of understanding the problem, planning, and computational skill are important. These three variables constitute the first three phases of Polya's (1973) four phases in the solution of a problem. These phases, as well as Polya's fourth phase, *Looking Back*, will serve as the basis for the suggestions for instruction in problem solving that will follow.

### Instruction Must Emphasize Understanding the Problem

One of the most neglected aspects of problem solving is that of *understanding the problem*. As Polya (1973) noted, this aspect of problem solving deals with far more than simply comprehending what is read. Understanding the problem implies grasping the relationships among the conditions of the problem and perceiving what is given mathematically. If a student truly understands a problem, he or she will not only be able to determine what is being sought, but will also recognize if the information given is reasonable or if a solution is impossible with the conditions as given. Reports of many studies indicate that much of the difficulty students have with problem solving stems from their failure to understand the problem.

In reporting the results and implications of the second National Assessment in Mathematics, Carpenter and others (1980b) noted that the multi-step and nonroutine word problems were difficult for all age groups. They noted that "the high levels of incorrect responses seem to indicate that there was little attempt to think through a problem in order to arrive at a reasonable answer" (p. 44). Moreover, in studies undertaken in connection with the Mathematical Problem Solving Project (Lester, 1975) it was found that many students often misread and misinterpreted problems and had difficulty in retaining and coordinating multiple conditions in a problem.

Several examples from the second National Assessment of Educational Progress (Carpenter and others, 1980a) further illustrate the difficulty students have with understanding the problem. One exercise on the assessment required finding the number of buses, each holding a certain number of passengers, that would be needed to carry a given number of people. Thirty-nine percent of the 13-year-olds gave responses indicating they ignored the fact that the number of buses must be a whole number.

In another exercise students were asked how many baseballs would be *left over* if a given number of balls were packed 24 to a box. A large percentage of students gave the quotient rather than the remainder as their response. As the authors of the interpretive report note, "Performance on these and many other exercises indicates that for too many students, problem solving involves little beyond choosing an operation, calculating an answer and reporting an answer" (p. 428). This comment is substantiated by the results of exercises in which extraneous information was given in the problem statements. Large percentages of students simply used all numbers given in some way, indicating a lack of understanding of the problem. When exercises with missing information were given, students also had a great deal of difficulty. In one such exercise, students were asked what else one would need to know in order to solve the problem. Fifty-six percent of the 13-year-olds and 32 percent of the 17-year-olds responded, "I don't know."

As was noted, the assessment results and the results of other studies indicate that many students read a problem and begin to manipulate the numbers in some way—often irrationally. Instruction should include slowdown mechanisms to motivate students to make a concerted effort to *understand* what is being asked. One such procedure is reported by Kalmykova (1975). Students read problems with inflection and try to convey meaning to one another. Then, students set up very simple problems to develop early habits of trying to understand a problem being presented.

Understanding a problem often involves translating it into some other form—a diagram, an equation, a matrix or a model. The important emphasis in instruction should be that the translation follows understanding, or leads to deeper understanding of the problem and not simply be a rote behavior.

Students in the elementary school are often taught to change "word sentences" into "number sentences." For example, in the problem

> The two fourth-grade classes at Longwood School are saving boxtops for athletic equipment. The need 10,000 boxtops to get a jungle-gym set. One class has collected 3,871 tops and the other class has 4,106 tops. How many more tops do they need?

One possible translation might be:

$$\begin{aligned}
\boxed{\text{NEED}} &= 10{,}000 - \boxed{\text{HAVE}} \\
\boxed{\text{HAVE}} &= 3{,}871 + 4{,}106 \\
\boxed{\text{HAVE}} &= 7{,}977 \\
\boxed{\text{NEED}} &= 10{,}000 - 7{,}977 \text{ tops}
\end{aligned}$$

To be able to make the translation, the student must first understand that what is needed is the difference between the total both classes have and the number needed, and will set up the translation accordingly. Students who do translations for problems such as this one by rote will often focus on the words "how many more" and set up a translation such as 4,106 − 3,871 = □.

Students should also be aware of the importance of understanding. In another of the Soviet studies, Gurova (1969) found that pupils' awareness of their own processes while solving problems had a positive effect on problem solving. One technique, which could be used in instruction to help students understand a problem, is to pose problems with missing information. This makes the problem impossible to solve. Another technique is to assign problems with redundant or contradictory information.

### Instruction in Planning

A second suggestion for instruction deals with helping students to construct *plans* for approaching problems before they begin to "attack" a problem. It is becoming increasingly clear that planning strategies are essential to arriving at correct solutions. Many researchers (such as Lester, 1975; Kantowski, 1975) have found that prior to instruction, many students do not appear to use any strategies in problem solving or use some form of random trial and error. However, instruction in planning techniques appears to have positive effect on the use of planning strategies and, consequently, on improvement in problem solving. Moreover, a variety of plans for the same problem should be suggested to emphasize to students that there is more than one way to solve a problem.

### Emphasis on a Variety of Solutions

Several studies, on both the elementary and secondary levels, indicate that students exhibit a great variety of problem-solving styles and preferences for certain ways to go about solving a problem. Moser (1979) studied children's representations of certain addition and subtraction problems with some emphasis on the students' interpretation of the problem structure and their imposition of structure on the given problems. One of the interesting features of the research findings was that even children in the first grade have a rich repertoire of strategies that they apply to a problem based on their interpretation of its meaning to them. For example, children who have not yet been taught an algorithm for subtraction approach simple "take away" problems in a variety of ways. Some count to the higher number on their fingers, then move backwards the desired number of places to get the result; others use a "counting up" procedure, beginning with the smaller given number. If concrete aids are available, some children set up a 1 : 1 correspondence and count the unmatched items to find the answer, while others use an "add on" model, starting with the smaller given number and finding the number needed to be added on to give the desired result.

In a study of nonroutine problem solving among secondary school students, Kantowski (1980) found evidence of a variety of styles in a given group. In the instruction used in the study, several solutions were given for each problem, some more elegant and efficient than others. The solution paths selected by the students varied widely. They did not always select what would be considered efficient solution paths, but those they followed suited their styles. Often, as problem-solving ability developed, students moved to more elegant solutions for problems similar to those they had solved earlier. Instruction should include a variety of solutions to problems to appeal to the variety of styles that might be present in a given group.

## Hints or Cues for Solution

One of the most frustrating aspects of problem solving for teachers and students alike is its all-or-nothing aspect. There is nothing more annoying than being unable to find the *one step*, formula, or piece of information that would unlock the solution to a problem. Students can be helped to move closer to a solution even when they are unable to find one piece of information. The Soviets have an interesting concept in the "zone of proximal development." It is a zone in which a student is able to operate *with assistance*, but in which he or she is not able to operate alone. The application of this concept to instruction in problem solving could be valuable. Students can be provided with hints or cues that could be used or ignored during the solution of a problem. Such cues could be useful to students who are at a dead end and for whom the solution would "open up" if a cue or hint were provided. In the problem involving boxtops stated above one "cue" might be the question:

> What is the total number of boxtops *both* fourth grades have?

Another "cue" could be the first translation:

$$10{,}000 - \boxed{\text{HAVE}} = \boxed{\text{NEED}}$$

and the question:

What is the value of $\boxed{\text{HAVE}}$ ?

For more complex problems, hints might include formulas (such as those for area or perimeter) or strategies that might be useful (such as guessing using a small number). Such "cues" or "hints" can also aid the students in understanding the problem. Students might be motivated to persevere, which could also result in more positive attitudes.

### Relationship of Problem Solving to Proficiency in Basic Computational Skill

In recent testimony before the U.S. House of Representatives, Ed Esty observed that "in general, it appears that what is being taught is being learned. We see this on the satisfactory performance on the lower level skills and, unfortunately, in the drop in performance on the high level problem solving skills" (*NAEP Newsletter,* February 1980). Furthermore, too much emphasis on computational drill may be counterproductive to development of the flexibility needed for problem solving. How can this demand for emphasis on basic skills be reconciled with the need for development of problem-solving ability in the limited time available for mathematics instruction? Results from the USMES evaluation indicate that students who engage in problem-solving experiences do not suffer in basic skill development. Students in experimental groups did at least as well on tests of basic skill (Shann, 1976). The Dutch look at basic skill from an interesting point of view. They distinguish between skill and mechanical practice and contend that "anyone who can execute standard routines but is incapable of solving a new problem *has no skill*" (Van Dormolen, 1976). Problem solving is a basic skill.

Applications for use of algorithms must be included when the algorithms are taught. *Selection* of an algorithm is one of the difficulties hampering effective problem solving (Kulm and Days, 1979). If applications were taught along with algorithms, this difficulty might be alleviated.

### Need for Changes in Evaluation Practices

Students are very *product*-oriented as is obvious from many of the studies reviewed. They are concerned with the *answer* and whether that answer is right or wrong. In several cases, it was seen that *understanding, setting up the problem,* and *planning* were factors in successful problem solving. Evaluation procedures should take into account these processes in order to motivate students to develop them. The curriculum must include some emphasis on understanding and planning. Exercises should reflect these emphases and teacher-made tests and grading procedures should give credit for using these processes. Questions such as "What do you need to be able to continue to solve this problem?" would make students think about what to do next and would assure students partial credit for understanding what to do instead of not giving them any credit at all if a computational error was made somewhere along the way. Students often work for grades, therefore grading practices should reflect all aspects of problem solving, including use of correct processes during solution.

## What the Curriculum Holds for the 80s in Problem Solving

Thus far, we have looked at some implications of research in problem solving for the teacher who is trying to translate some of what has been found to classroom practice. But research, both educational and technological, has far-reaching implications for curriculum developers in addition to the suggestions for instruction cited.

Conspicuous by its absence in the discussion to this point has been the role of the calculator and, more dramatically perhaps, the role of the microcomputer in the problem-solving curriculum of the 80s. One reason for this, of course, is that it is too soon to see many published research studies dealing with the microcomputer. That does not, however, mean that work is not being done in problem solving using the computer. Of those 61 presentations in problem solving at NCTM's 1979 National Convention, several included instructional techniques using the microcomputer.

Teaching for problem solving is one of the most difficult tasks facing the teacher at any level. Let us consider for a moment some reasons for this difficulty.

(1) Often there are no new concepts to introduce or algorithmic skills to teach—the object of the instruction is to have students put together knowledge they have already acquired to solve the given problems, and techniques to do this are not readily available.

(2) All students in a given group are not familiar with the necessary content or the algorithms needed to solve some of the problems encountered, and the teacher is then faced with the problem of how to handle the diversity of backgrounds.

(3) That students work at different rates is especially true in the case of problem solving. Some students need much more time than others to understand a problem and to find what is being sought.

(4) There are many problem-solving styles resulting in different paths to the solution of problems, particularly those that are nonroutine (Moser, 1979; Kantowski, 1980). It is difficult for a single teacher to take into account the variety of styles that occur.

(5) Teachers are faced with increasing requirements in the curriculum and are pressured to emphasize computational skills and so have little time to assist students who are having difficulty in problem solving experiences.

(6) There is a lack of good sequences of related problems to use in instruction.

Many of these problems can be partially resolved through the use of computers. The computer brings the potential for radical change in the mathematics curriculum and for great support for teachers, bringing some relief in overcoming the difficulties in problem solving instruction outlined above. The microcomputer is not a panacea. It cannot resolve all the difficulties encountered in an effort to teach for problem solving, but it can definitely provide support in many of the problem areas. Specifically, a student interacting with a microcomputer can work at his or her own pace and take as little or as much time as needed on a given problem. The branching capability of the computer is perhaps the single aspect of this machine that makes it such an invaluable tool in teaching for problem solving. It can enable a student to request hints or cues or other information such as an algorithm that may have been forgotten or some instruction in content that was, perhaps, never learned. More important, possibly, this branching capability can take into account the many preferences or styles encountered in any group, by permitting a student to select a desired path to solution and to request cues to aid in finding a solution in his or her own preferred style. The capacity of the computer to provide for differences in educational backgrounds and preferred styles enables the teacher to deal effectively with what could otherwise be an unmanageable situation. Moreover, the capability of the computer to collect and process data can be used to provide feedback to a teacher on many aspects of a student's problem-solving experience. For example, a record of paths followed, hints or cues selected, as well as information needed (algorithms, instruction in content) can give a teacher a valuable profile of a student. Software to handle such demands of instruction for problem solving will be an important demand in curriculum development for the 80s.

The graphics mode of the computer cannot only provide excellent color diagrams related to problems, it can also simulate motion in a way not possible on the printed page or even in other media. This capability of the computer has implications for curriculum development as well as for further research related to spatial abilities. It has been observed in working with students (Kantowski, 1980) that the complexity of a figure will often keep a student from finding a solution to a problem. If the figures can be seen as they are generated, perhaps their complexity will be less overwhelming, and a solution more readily found.

The availability of the calculator mode of the computer also makes it an invaluable tool in teaching for problem solving. Conceptually simple problems previously unsuitable for widespread use because of tedious and time-consuming calculations may now be included at relatively low levels of instruction. Moreover, real problems that could often not be studied because of the tedious calculations required can and should be included in the curriculum at all levels. For example, in a recent exploratory study, I found that students were able to solve very complex problems in number theory with the aid of a microcomputer. In subsequent interviews with students after the problem-solving sessions, many admitted that they never would have attempted to solve some of the problems without the aid of the computer (Kantowski, 1980).

## Need for Problems to Use in Instruction

Because students learn by solving similar kinds of problems (Polya, 1973; Cambridge Conference, 1963) sets of related problems need to be developed. Such sets could include problems of similar mathematical structure, problems involving similar content, or problems for which similar solution techniques would be useful. This reiterates the call for sequences of problems made by the participants of the Cambridge Conference. Although almost two decades have passed since the publication of their report, their belief that "the composition of problem sequences is one of the largest and one of the most urgent tasks of curricular development" (p. 28) is still relevant today.

In summary, the future looks very bright for the teaching of problem solving in the curriculum of the 80s. During the last decade, we have begun to see new trends for further research and directions for curriculum development suggested by the studies of the last few years. The curriculum promises to be more child-centered if new technology can be used to advantage in dealing with the diversity of problem solving styles. And, after all, the learner is what we are all about. Success in problem solving is a very rewarding experience—finding clues to so much lack of success in the past could be the breakthrough we've been seeking. The hope for the curriculum of the 80s is that the problem-solving component will present a new perspective to teachers and students alike. The prospect of new and more challenging problems and the applications promised by the advent of computer technology give us assurances of an exciting decade.

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# Response

## Larry K. Sowder

Kantowski's chapter is especially important because of the likely forthcoming re-emphasis on problem solving in the mathematics curriculum. She has done an excellent job of clarifying how researchers use the term "problem," of describing research trends, of identifying problem-solving components, and of suggesting some research implications for instruction and curriculum development. I heartily endorse the spirit of her remarks.

## The Problem-Solving Components

**Understanding the Problem.** During this first phase of solving a problem, a student should be able to call on a firm conceptual basis. For example, consider a student whose "understanding" of fractions is confined to rules of computation. The student is certain to be handicapped in dealing with a verbal problem in which a fraction appears, unless some rotely-learned procedure happens to fit the context of the problem. Studies which have manipulated the amount of time devoted to concept development have found that *at least 50 percent of class time should be spent on concept development* (Shipp and Deer, 1960; Shuster and Pigge, 1965; Zahn, 1966). In our example, a fraction should not be allowed to be only two numerals separated by a bar. Nor should a student's concept be limited to only one kind of model, like pie shapes. Fractional number concepts must be firmly founded in several models like folded papers, pieces of string, colored rods or strips, and sets of objects, for example, as well as work with circular and rectangular regions and number lines. The student would then have a richer conceptual basis from which to draw when confronted with "¾ of the distance" or "¾ of the children" or "¾ of the amount" in a verbal problem.

In addition to the anecdotal evidence cited by Kantowski, some research studies suggest that students can profit from reflecting about problems. For example, middle schoolers perform better on problem solving if allowed to suggest possible ways to solve a problem, to discuss these ways, and then to arrive at a consensus (Blomstedt, 1974). Such a procedure contrasts sharply with the common modeling practice in which the teacher "shows" the students how to solve a problem. In the same vein, Rowe (1969) reported that teachers wait only about one second after asking a question before calling on a student, and then again only about one second after the student response before proceeding to the next question or remark. How much time does that allow for reflection? She found that the quality and quantity of student responses increased significantly when teachers sought to lengthen their wait-times to five seconds. Surely such results have implications for our questioning during problem solving! Finally, at least two studies (Graham, 1978; Keil, 1964) suggest that students can profit from making up their own problems and solving them. It seems clear that teaching procedures other than "monkey see, monkey do" are called for if our students are to develop a deeper understanding of a problem.

**Planning.** For the routine elementary school verbal problem, planning often comes down to the selection of the proper operation(s). The points in the previous section, especially the desirability of a strong conceptual basis, are directly related to one's ability to decide whether to add, subtract, multiply, or divide.

Kantowski mentions the use of cues or hints for solution of a problem, but this advice may be misinterpreted. For example, some teachers tell students to look for "key" or "clue" words which can suggest what operation to perform. Such advice is well-intended and sound insofar as the student then thinks about how the variables in the problem are related. The advice, however, can be misapplied by children. Consider a student who has been told that the word "gave" signals subtraction. If the student thoughtlessly applies that "rule" to this problem: "Pat had 50¢. Then Grandma gave Pat 35¢. How much did Pat have then?"—the student will not get the correct solution. Note that this thoughtless application is possible if the understanding-the-problem component is bypassed. Cues and hints for solution must not be used to circumvent thinking about the problem. Even first-graders blindly apply the key-word approach if they have been taught it (Nesher and Teubal, 1975).

**Carrying Out the Plan.** The solutions of most routine verbal problems and of many nonroutine problems involve computations. A quote from the NCTM agenda for the 1980s is appropriate:

> It is recognized that a significant portion of instruction in the early grades must be devoted to the direct acquisition of number concepts and skills without the use of calculators. However, when the burden of lengthy computations outweighs the educational contribution of the process, the calculator should become readily available (p. 8).

We must not delude ourselves by thinking that inserting longer or more frequent lists of story problems into a textbook is in itself the way to insure a focus on problem solving, if the lists do not require understanding the problem and planning. If problem solving *is* to be the focus, then these components, rather than computation, should receive the emphasis during work on story problems. It is consistent with such an emphasis to leave at least some of the computation connected with verbal problems to calculators.

**Looking Back.** Looking back—reviewing the solution of a problem just solved, seeking other possible solutions, and thinking of other problems that could be solved in the same way—is a component of problem solving that has not been subjected to much research even though it often appears in lists of problem-solving advice (Polya, 1971). Looking back would offer another chance for students to reflect on the problem and its solution. Despite the urgency often felt in the classroom to move on to the next problem immediately after solving one, a minute or two spent on looking back could allow attending to the processes involved rather than leaving the impression that the sole concern is the answer.

### Research and Development

The NCTM agenda at least reminds us that the horse belongs before the cart. Whether the horse remains obscured by the cart remains to be seen. A great challenge to curriculum developers will be to make it possible for busy teachers of the 1980s to focus on problem solving. Supervisors will also have a great challenge: convincing some teachers that such a hard-to-teach emphasis is the proper focus. Kantowski has been perhaps too sanguine in making a case for microcomputers as a prime problem-solving vehicle. Certainly microcomputers in the schools are easily justified on a computer literacy basis; software to accomplish the many things that Kantowski envisions, however, is in an infant state. (Indeed, Kantowski is engaged in pioneering work using the microcomputer as an aid in problem-solving instruction.)

One difficult area of research deserves more attention: affective factors in problem solving. We have all observed even young children work concentratedly on an occasional task which has intrigued them. It would seem natural, then, to try to identify intriguing tasks and to isolate what makes them intriguing. Researchers have tried to tie verbal problems to student interests but by-and-large have not found that approach to yield better performance (Cohen, 1976; Travers, 1967). Perhaps having students write their own problems gave the better results (cited above) because of some affective-cognitive interplay.

It *is* encouraging that so many researchers are giving attention to problem solving, routine and nonroutine. With one and a half million "mini-laboratories" in operation every school day, with teachers trying different things, often on a hunch basis, one can still wish for a coherent, concerted research approach. If every school district were to seek out or develop a study of some aspect of problem solving, perhaps our teaching of problem solving could proceed on a more scientific basis.

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# VIII. Computers

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Mathematics instruction in elementary and secondary schools is frequently perceived to be more amenable to the use of computers than are other areas of the school curriculum. This is based on the perception of mathematics as a subject with clearly defined objectives and outcomes that can be reliably measured by devices readily at hand or easily constructed by teachers or researchers.

Because the purpose of this book is to provide implications of research evidence rather than a historical review of research, the studies include those that have been undertaken, completed, or published during the decade of the 70s. Vinsonhaler and Boss (1972) summarize ten major studies of CAI drill and practice that were completed prior to those reviewed in this chapter.

As with other chapters in this book, the literature reviewed here is limited to that which is clearly research. Some related topics otherwise of much interest to teachers and curriculum developers are omitted. Included, for instance, are few descriptions of current CAI programs, little identification of problem-solving modes for which computers are being used, and no descriptions of individualized instruction programs that are being managed by computers. A number of resources are available elsewhere for the interested reader. Wang (1978) lists nearly 3,000 programs being used throughout the United States. Bukoshi and Korotkin (1976), reporting on computer activities in secondary education, indicate that in 1975, 58 percent of secondary schools used computers for administrative or instructional purposes. This is an increase of 34 percent since 1970. Forty-three percent of the 1,459 computer-based courses were in mathematics.

Even though schools seem to be organized as though learning occurs in compartmentalized fashion, we know that *even* with mathematics, a learner's self-concept, ability to read, and interest and functioning in the social and scientific world that surrounds us not only influence learning, but provide an integral part of growing up during the elementary and secondary school years. Much of what we continue to learn about reading, for instance, has implications for mathematics instruction; what we know about the use of computers with adults, as in military research, must have implications for mathematics; and the vast literature on the technology of mass media must also have implications for mathematics. Yet all of these sources of information are to go untapped in this chapter.

## Computers

Computer technology used in mathematics education can be divided into two categories. The first of these is computer-assisted instruction (CAI), which, in turn, can be divided into drill-and-practice programs, instruction in mathematical concepts, problem solving, and computer programming. The second use of computer technology is in computer-managed instruction (CMI).

### Computer-Assisted Instruction

The Computer Curriculum Corporation (CCC), under the direction of Patrick Suppes, and the mathematics instruction program included in Programmed Logic for Automated Teaching Operations (PLATO), under the direction of Robert Davis, represent the most extensive CAI programming and research efforts in mathematics education underway during the 70s. There are some interesting and significant common characteristics of these two comprehensive and visible programs. Both programs represent long-term developmental efforts by their respective directors. Suppes and Davis were mathematicians who turned their attention to school mathematics instruction early in the 60s; they were heavily involved in the development, implementation, and dissemination of mathematics programs for the schools during the period of the new math. Both worked extensively with the schools and directly with children within those schools. Both developed school program materials that were supported initially by research and development funds, with later editions of the materials made available to schools through commercial publishers. Suppes and Davis drew heavily on these earlier experiences and materials in the development of the courseware that has become central to their CAI efforts. A central point that must be made about these two programs, then, is that the content or substance of these CAI efforts has a substantial developmental history.

There are a number of other important points to be made about the two programs. They function on two of the largest computer systems committed to instructional use. Suppes has made interactive satellite transmissions of programs to South America and to Eastern states of the U.S.A. from his center in Palo Alto, California. The work Davis has done at Illinois uses only a small portion of the PLATO system for his elementary mathematics program. Suppes and Davis have both accompanied their instructional efforts with research efforts designed to monitor the effectiveness of their programs. Suppes has been at work with computers longer than Davis and, therefore, has a much larger body of research support for his efforts. In addition, Suppes' evidence includes data from a wider variety of populations than does the Davis research. Although both programs are intended to supplement the work of the regular classroom teacher, Suppes has focused largely on drill and practice, whereas Davis has included drill and practice along with other CAI instruction.

### The Computer Curriculum Corporation (CCC)

Suppes (1979) has reported that CCC courses include the largest number of students using CAI in this country. By 1978, that number was in excess of 150,000 students in 24 states. Most of these students were disadvantaged or handicapped. A description of these courses and the strands strategy they represent is contained in Suppes (1979) and in Macken and Suppes (1976). The content strands include number concepts, horizontal addition, horizontal subtraction, vertical addition, vertical subtraction, equations, measurement, horizontal multiplication, laws of arithmetic, vertical multiplication, division, fractions, decimals, and negative numbers.

The evaluation of the effectiveness of CCC mathematics instruction as implemented in 21 different sites is reported by Macken and Suppes (1976) and by Poulsen and Macken (1978). Most populations were largely disadvantaged or handicapped youth. Many were Title I students. Others were either deaf, low IQ, or minority students whose opportunities for previous schooling had been severely limited.

In general, the data in these several studies were analyzed to answer three questions: (1) How was time on the CAI terminal related to achievement gains? (2) How was gain in achievement related to expected gains? (3) How was CAI placement related to standardized test placement of students? Additionally, there was considerable evidence gathered concerning the satisfaction of parents, teachers, and students with their work in CAI.

There is consistent evidence to support the claim that achievement gains are related to the amount of time students spend in CAI. (Program developers insist that ten minutes per day, five days per week, for a total of 1,500 minutes per year is the optimum time students should spend in CAI. Though planning for that amount of time is part of the conditions for participation in the CCC program, there is much evidence that it is a seldom-achieved goal.)

The most extensive CCC study concerning the relation of time at the CAI terminal to achievement (Poulsen and Macken, 1978) included data from a number of schools throughout the southern half of California. The subjects were largely disadvantaged students in grades three through nine. The authors report that ". . . no group received more than 75% of the recommended time, and most groups received considerably less, even though most students were scheduled to receive ten minutes of CAI per day . . . " (p. 3). The school day schedule is continuously altered with extra activities that occupy children's time and interests. Time taken away from regularly scheduled class activities is as likely to be taken from CAI as it is from any other scheduled activities in the school day. The proportion of the 1,500 recommended minutes per year actually spent in CAI ranged from 23 to 75 percent among the 20 groups reported. Within grade groups from three through nine in each of seven schools, the correlations between time in CAI and grade placement gain ranged from .53 to .99. The average correlation for all schools was .86. Such a correlation suggests a very strong relationship between time in CAI and grade-placement or achievement in mathematics.

The ratio of actual to expected gain in mathematics achievement as measured by the standardized test then in use in each district was consistently high throughout the several studies reported (Poulsen and Macken, 1978). In Freeport, New York, the ratio of actual gain to expected gain was 1.54 for 142 students who were initially one year below grade level and 1.91 for 20 Hispanic students who had experienced little previous schooling. In a study in Isleta, New Mexico, 96 students were in CAI for a period of seven months. In each of the four classes, the average gain was more than one month for each month in CAI; and the average growth rate was 1.33 months per month. The mean gains for the year made by Title I students in CAI at Shawnee Mission, Kansas, as measured by pretests and post-tests with the Key Math Diagnostic Arithmetic Test, ranged from .99 in grade two to 1.77 in grade five, with an average gain of 1.41 across grades one to six. There is, then, in these and other reported studies (Macken and Suppes, 1976; Poulsen and Macken, 1978), ample evidence that the CAI program developed and implemented through CCC consistently obtained results that surpassed expected gains for students least expected to succeed in mathematics as a result of regular classroom instruction.

Of specific relevance to the CCC mathematics curriculum is the question of the relationship between CAI placement, as determined by the student's performance in the instructional program, and standardized test placement. These studies reported consistently high correlations between these two measures. Among the 20 groups reported in the southern California study, the correlations between CAI final grade placement and the California Test of Basic Skills (CTBS), placement ranged from .28 to .87. It should be noted, however, that the next lowest correlation after .28 was .53 and the average of all correlations was .74. In the Ft. Worth study, correlations between CAI placement and the Stanford Achievement Test (SAT) mathematics computation grade placement were .62, .51, and .56, respectively, for grades three, four, and five (Macken and Suppes, 1976). The correlations between CAI placement and SAT mathematics applications grade placements were .35, .52, and .59 for grades three, four, and five. There does, then, seem to be a relatively high correlation between CAI placement and placement on standardized measures. These findings indicate that achievement on CAI mathematics as defined and developed by CCC is an appropriate curriculum for the development of skills and concepts measured by standardized instruments such as the Stanford and California achievement tests.

Though there has been a less direct attempt to measure the affective impact of CAI in the CCC program, a number of reported comments seem pertinent. Crandall (1977) reports that the CCC program in Los Nietos, California seemed to reduce truancy and vandalism in his school. St. Aubin, reporting on the Dolton, Illinois CAI program for the handicapped (see Macken and Suppes, 1976), claims that using the computer for individual work resulted in students' improved perceptions of themselves and their school. He reports that after some initial hesitancy, teacher response to CAI was positive and exciting as they saw improvement in the children's self-image.

To summarize the research findings of the CCC program, we have positive results for each of the questions for which answers were sought.

(1) Time children spent at CAI terminals was positively related to their achievement.

(2) There was substantial evidence that actual achievement gains exceeded expected gains based on previous experience of the subjects.

(3) There was evidence that grade placement as determined by the CAI program was highly correlated with grade placement on standardized tests.

(4) There was much subjective evidence to support the claim that attitudes of students and teachers toward CAI were positive.

### PLATO and Elementary School Mathematics

Another CAI program designed for use in the elementary school (grades four through six) is the elementary mathematics program developed over a number of years by Davis and modified for use with the PLATO computer on the University of Illinois campus at Champaign-Urbana. Sixty interactive PLATO terminals are dedicated to elementary school mathematics. Four of these terminals are present in each participating classroom. "Each student received ½ hour of mathematics lessons, via computer, each school day, plus whatever instruction the teacher chose to provide. In fact, each teacher continued the 'regular' math curriculum from pre-PLATO years, except that a few teachers made adjustments to help relate the 'regular' curriculum to the PLATO curriculum" (Davis, 1980b).

The four content strands of the PLATO program were derived from Davis' earlier work. The first three strands are viewed by Davis as representing, respectively, content that is usually taught successfully in the schools (whole numbers); content that is not so successfully taught in the schools (fractions); and content that is not usually taught in grades four through six (graphs and functions). The fourth strand, concerned with programming computers, provides an option for children as a fringe benefit, but is not viewed as a part of the demonstration-research project.

It should be recognized that the PLATO project is not one that follows a typical programmed instruction format. Rather, PLATO mathematics is presented via terminals that include both the teletype and an audiovisual interface between computer and student, making it possible to follow the general instructional formats used in the earlier Madison Project materials. Davis calls the Madison Project strategy *paradigmatic learning experiences*.

The major research to determine the effectiveness of this instruction was undertaken during the 1975-76 school year by Swinton, Amarel, and Morgan (1978). Students in a dozen classrooms using the computer program were matched with students in classrooms that did not use the computer. At every grade level, children using the computer made significantly greater achievement gains than children in the control group on measures associated with the program as well as on computation and applications subtests of the California Test of Basic Skills. Of importance also were the positive attitudes exhibited by children.

Paraphrasing the final report to NSF (Slottow and others, 1977), Davis (1980b, p. 9) speaks of the strongly positive reactions of students and teachers to the PLATO mathematics program:

> On every single attitude question used, differences strongly favorable to PLATO were observed. Pupils were enthusiastic about the mathematics lessons which the computer presented on the TV-like screens, may students sought *extra* sessions, their attitudes toward mathematics improved (as measured by a questionnaire), and so did their attitudes toward their own ability to deal with mathematics. Teacher assessments, though inevitably subjective, were very strongly positive, including even reports that PLATO had decreased anti-social behavior.

A few children's quotes taken from Stake's report (1978) of PLATO and fourth-grade mathematics illustrate the informal relationship children have with their computer teacher:

Dear Plato,
Why does PLATO get messed up a lot?
From Cool Cat

Dear Plato,
This was a very nice session. Not too hard or too easy.
I am glad someone was able to invent you.
Sally R.

Dear Plato,
I like the games you play. But now I have to go.
Kitty N.

Swinton, Amarel, and Morgan (1978, p. 24) report:

> A particularly important outcome was revealed in positive effects on instruments designed to measure students' understandings of any ability to represent concepts and operations, beyond mere facility in manipulation of symbols. The PLATO system here demonstrated that it was capable of *teaching*, as well as of providing drill and practice of concepts already introduced by classroom teachers.

That teachers and the PLATO program are important complements to one another cannot be denied. Swinton and others (1978, p. 25) conclude that "Teacher effects are real, large, and idiosyncratic." The PLATO mathematics program is not teacher proof; it is not independent of the decisions and actions of individual teachers. Rather, the PLATO system is experienced differently by children in classrooms of different teachers. The authors report that teachers perform most effectively when they are given control over the curriculum. Though this is the case in many uses of computers in mathematics instruction, it is more apparent in PLATO than in Suppes' CCC program, where children are scheduled at CAI terminals out of the classroom and management diagnosis and prescription decisions are designed in the program.

PLATO mathematics was being developed in a number of dimensions at the same time, and Swinton and others (1978, p. 25) warn that

". . . simultaneous system and curriculum development is hazardous. . . ." They make a point of suggesting that more attention needs to be given to the development of courseware prior to research efforts than is frequently the case, and that it probably does make a difference who is involved in the development work. These researchers express a preference for those persons deeply involved in the subject matter, with extensive teaching experience and with a proven track record in curriculum development work, over those persons whose first interest is in the computer and who then seek a subject matter in which to make an application of computer expertise. The authors conclude with an expression of support for continued developmental efforts in the PLATO project and describe the system as having ". . . demonstrated its potential as a curriculum test bed . . ." (p. 26).

### Comparative Research Studies

In addition to the major research and development efforts of Suppes and Davis, there are a number of other projects that have been reported in the literature during the 70s.

Nine studies comparing achievement and attitudes of students using computers with noncomputer students are summarized in Figure VIII-1. No researcher is represented more than once in this list, indicating that research efforts appear to be isolated. A study of student attitudes by Hess and Tenezakis (1973) is an interesting example included among these studies. In the study, there were 189 seventh to ninth grade subjects, 50 of whom had taken the CAI drill-and-practice program with the Suppes CCC materials for a period of one or two years. The 139 non-CAI comparison students were, on the average, performing better than the 50 CAI subjects, though among the CAI students were several who had been in the program because of their need for remediation.

Subjects were asked to compare their perceptions of computers with their perceptions of teachers and of textbooks. The CAI group indicated that computers had some real advantages over the classroom teacher. They viewed computers as fairer, easier, clearer, bigger, more likeable, and better than the teacher. The CAI group perceived the computer as having more information and making fewer mistakes than the teacher. According to the authors, students perceived computers as being more "charismatic" than teachers, with greater endurance of work, greater infallibility, and greater capability to help a student improve grades in mathematics. Non-CAI students viewed the computer even more favorably, perhaps reflecting the mystique of the computer that is so prevalent in society. In a further discussion of the effects of CAI instruction on students' perceptions of

**Figure VIII-1. Studies Comparing Achievement and/or Attitudes of Students Using Computers with Noncomputer Students**

| Name of researchers | Year of Report | Grade level | Program type | Achievement | Attitude |
|---|---|---|---|---|---|
| Crawford | 1970 | 7 | Drill & Practice | ns | |
| Hatfield and Kieren | 1972 | 7,11 | Programming | + | + |
| Street | 1972 | 3-7 | Drill & Practice | ns | −s |
| Martin | 1973 | 3,4 | Drill & Practice | +* | + |
| Hess and Tenezakis | 1973 | 7-9 | Drill & Practice | | + |
| Milner | 1973 | 5 | Programming | ns | + |
| Smith | 1973 | 7-9 | Drill & Practice | | + |
| Robitaille and Sherrill | 1977 | 9 | Programming | −s | + |
| Morgan | 1977 | 3-6 | Drill & Practice | s | + |

s = differences significantly in favor of computer students
−s = differences significantly in favor of noncomputer students
+ = differences in favor of computer students
− = differences in favor of noncomputer students
ns = differences not significant
* = fourth-grade boys and low-ability students achieved more than others.

teachers, Brod (1972) reports that especially during the first year of CAI instruction, involvement reduced students' dependence on the teacher for task-specific resources. This undermining of the teacher's authority, he suggests, represents an unanticipated and undesirable consequence of CAI instruction.

Thirty-two dissertation studies (1969-1979) reporting on the comparative effectiveness of CAI and regular classroom instruction in changing achievement and attitudes of students were reviewed. In most of the studies, CAI was used for drill and practice, although in five, the computer served as tutor; in seven, the emphasis was on programming; and in two, problem solving. In comparison with the research studies reported in Figure VIII-1, results among the dissertation studies appear to be less consistently positive both for achievement and for attitudes. Of the 30 studies that compared achievement differences, 18 reported no significant differences, whereas 12 did report some significant differences that favored the

CAI groups. Of the 13 studies that compared attitudes, eight reported no significant differences and five reported significant differences (four for the CAI group and one for the non-CAI group). These dissertation studies as a group fail to generate support for a relationship between computer-assisted instruction and the attitudes of students. Though the achievement picture was varied, nonsignificant studies outnumbered those that significantly favored the computer group by 18 to 12.

Clearly, there are many problems with any attempt to summarize the findings of comparative studies of CAI. The unknown quality of the CAI instructional components (courseware) and the certain unevenness of that quality raises many questions about individual studies and about the results as a group. The hardware/software configurations are also varied from study to study. Indeed, there are a number of instances in which the still-developing state of either the hardware, the software, or both created problems that clearly had an impact on the results of these studies. The nature and extent of teacher involvement in planning for the supplementary CAI instruction differed from study to study and may have been quite minimal in most or all of the studies in this group. The lack of teacher involvement early in planning and implementation of computer use in classrooms would seem to be a major problem in many current computer applications. Nonetheless, there seems to be only minimal evidence from these studies that one could confidently proceed with such CAI programs in contexts, such as those used for these students with the expectation that achievement will be improved.

### Additional Studies of CAI

A few studies have been reported that have investigated specific aspects of CAI, but have not been concerned with the comparative results between students with and without CAI instruction as in the case of studies included in the previous section.

Taylor (1975) reports on adaptive mastery, typical mastery, and traditional nonmastery models which employed different criteria for terminating CAI practice. All seventh-grade subjects received instruction in basic arithmetic skills. The adaptive mastery model differed from the typical mastery model in that it provided variable amounts of practice depending on feedback rather than a fixed amount of practice. Though the adaptive mastery model required less time, fewer practice items, and minimized overpractice, students in this group reached the same level of performance on post-tests and on delayed retention tests as did students in the other two models.

Keats and Hansen (1972) report a study in which they investigated the effectiveness of different kinds of CAI feedback. The area of study for 45 ninth-graders was proofs in mathematics. Feedback for the three groups of 15 students each included verbal definitions, numerical examples, or a combination of both types. The latter was thought to be more like that of typical classroom instruction. Though there were no significant differences in post-test scores, error analysis by groups over the 11 exercises revealed that feedback in the form of verbal definition was more helpful than providing the learners with numerical examples or with a combination of the two. The authors conclude that this finding supports previous research, and especially is in keeping with Ausubel's (1961) support for providing the learner with a verbal explanation of underlying principles.

In a study by Herceg (1973), top track and middle track algebra II students were assigned to three computer treatment groups: individual rate setting with formally presented objectives, traditional classroom setting with formally presented objectives, and traditional classroom setting without formally presented objectives. Top track students did not achieve significantly higher when they were provided objectives for the unit. Middle track students, however, did achieve significantly higher when they were made aware of the objectives of the unit, although those in the individual rate setting treatment achieved significantly lower than students in the traditional classroom setting.

Dienes (1972) investigated the pacing question as it relates to drill-and-practice computer applications with sixth–grade students. His study was divided into two parts. In the first part, all 167 students completed a part of the computer program at their own pace. On the basis of this self-paced experience, students were assigned to treatment groups for which the pace of inaccurate students was decreased while it was increased for slow and accurate students. Fast and accurate responders were assigned to a "task-mean" treatment. Control groups continued to proceed at their own pace. There were significant differences in the achievement of treatment and control groups. Such external pacing assistance was beneficial to those students who did not adopt appropriate pacing habits.

Alspaugh (1971) reports that high school students learned FORTRAN programming language as well as college students, although they required twice the number of hours of instruction. It was suggested that "the grade placement for beginning FORTRAN courses can be lowered from grades 15-16 to grades 11-12 with comparable achievement . . ." (p. 47).

The computer has been used frequently as a tool to investigate learning and teaching strategies. Such a study is that by Kraus (1980) in which he investigated the heuristics of problem solving as subjects played a computer version of the game of NIM.

### Computer-Managed Instruction

The 70s witnessed considerable activity in the development of computer-managed instruction (CMI) programs, though little research has come from those efforts. CMI systems provide a means for keeping information concerning available learning resources and the learning progress of individual students. In individualized instruction programs, CMI systems assist with the diagnosis and prescription of learning activities. Baker (1978) has provided an excellent review of such systems and the status of CMI by the mid-70s.

Several dissertation studies have compared the achievement and/or the attitudes of students who have experienced CMI with students in traditional classrooms. Of the six studies reported in Figure VIII-2, most report positive results that favor computer students over non-CMI students, though these differences are seldom significant.

**Figure VIII-2. Dissertation Studies Comparing Achievement and/or Attitudes of CMI Students with Non-CMI Students**

| Name of researcher | Year of study | Grade level | Achievement | Attitude |
|---|---|---|---|---|
| Miller, Daniel | 1970 | 6 | s | |
| Miller, Donald | 1970 | 6 | | s |
| Lee | 1972 | 5 | | + |
| Akkerhuis | 1972 | 6 | s | ns |
| Wilkins | 1975 | 8 | + | |
| Chanoine | 1977 | 4-6 | + | |

+ = differences in favor of computer students
s = differences significantly in favor of computer students
ns = differences not significant

## Implications from Research in Computer Applications

The research of the past decade has been conducted on programs using the facilities of large-scale computers. As a new decade begins, the microcomputer is clearly seen as the way of the future. The flexibility of these computers, the control of these microcomputers at the local level (even at the classroom level), and the potential for involving students and teachers in a wider range of computer problems and technologies place issues raised during the past decade into new contexts. In these new contexts, new questions will be asked, many of which differ substantially from those raised in the immediate past. Nonetheless, a number of implications can be drawn for the future from the research that has been identified here. On the one hand, it appears that studies investigating computer technologies in mathematics instruction are too few, too piecemeal, and too unclear in their results to provide certain direction for elementary and secondary education. On the other hand, however, there does seem to be evidence emanating from specific centers that provides positive expectation from technology for the decade of the 80s.

*Consistently positive results appear to come from centers that have a long, dedicated history of effort directed toward mathematics instruction.* Results from the Stanford work of Suppes and that of Davis with PLATO at Illinois provide substantial evidence that CAI can be consistently effective in mathematics instruction and that such instruction even now is within reasonable cost bounds. During the 80s application of microcomputers to instructional tasks may be expected to greatly reduce costs compared to those incurred by the use of larger system computers that have been available during the 70s, and on which most or all research reported here was conducted.

Inconclusive results tend to accompany those projects which are associated with short-term efforts or are in their first two to four years of operation. Most of the dissertation studies appear to be a part of this set of short-term effort. No doubt many of these dissertation studies will provide the experience and insights on which the trends in the late 80s will be based, but few are sources of research evidence that can be used to support technological applications in mathematics instruction at the current time.

*Mathematics learning by disadvantaged youth can be improved through certain computer applications of CAI.* Studies that have investigated the impact of CAI on the mathematics learning of Native Americans, the deaf, inner-city Blacks, and bilingual Spanish-speaking youth have shown that mathematics achievement can substantially exceed expectations based on previous experiences. There is currently less evidence that "average" or "above average" learners can be helped as much. Federal, state, and foundation funding has been much more available for disadvantaged subgroups of children than for the general population. This has encouraged researchers to seek funding for hardware, software, and courseware development for applications to these special groups; hence, our current evidence provides more information about such learners than we have for others.

The history of educational innovation during the past two decades has been one that has taught us the importance of homegrown products. However, with the cost and time required for technological innovation, *we may be at a time when local districts will find it necessary to draw more* *heavily on the successful products developed and demonstrated effective elsewhere.*

Clearly, the development of computer-assisted instruction programs for use in the schools is in its infancy. The integration of computer activity with other activities underway in the classroom or with other activities that are particularly feasible because of the computer technology is only now beginning to be developed. *Needed are CAI programs that lead learners away from the computer and make it possible because of the computer to explore the world in ways not otherwise possible.* The addiction that accompanies those who pursue computer programming professionally must not be encouraged by the manner in which computers are used for instruction in the decade of the 80s.

There is some evidence that students and teachers react favorably to the use of CAI or CMI. Attitudes can appear positive even in instances where the results of achievement data are indifferent. It would seem, therefore, a fair warning that *attitudes alone may not be an adequate measure of the effectiveness of computerized classroom instruction.* The minimal evidence suggesting that the introduction of computers in the classroom, at least initially, reduces students' perceptions of the teacher's authority has considerable implication for the preservice and inservice preparation of teachers. *High technology introduced into the classroom must be brought by the teacher, not by an outsider to whom the teacher must turn each time a question arises concerning the use of that technology.* With the advent of microcomputers, teachers need relatively minimal preparation to take charge of those computers in the classroom. As an extension of the teacher who is competent to operate the computer and who understands how it may be best integrated into the classroom curriculum, technology can be used to sustain or boost the authority of and respect for the teacher in the classroom. When teacher and technology are seen as separate components of the classroom environment, humanism and technology also become separate environments. DeVault and Chapin (1980) have pointed to the need for a balance between supply and demand as the supply of microcomputer technology becomes more readily available through technical advances and cost reductions. The demand side of the supply and demand equation rests in the hands of the teacher, and technology must respond to teachers' perceptions of classroom instruction. Throughout development and implementation efforts, teachers must have roles that are comparable in importance and impact with those of the technologists if the introduction of this technology is to succeed in the classroom.

Certainly much of the adverse criticism of technology in the schools comes from fear that many humanistic characteristics of elementary and secondary schools will be sacrificed for the perceived or actual efficiency and effectiveness of technology. Much of our concern for humanism is centered around the search for better ways to meet individual differences and to assist students in reaching their full potentials.

Suppes (1979) alluded to these humanistic values:

> We do not yet realize the full potential of each individual in our society, but it is my own firm conviction that one of the best uses we can make of high technology in the coming decades is to reduce the personal tyranny of one individual over another, especially wherever that tyranny depends upon ignorance.

His own research and that of others investigating the effectiveness of his CAI programs indicate that those learners who may be expected to be low achievers in terms of their past histories can be helped through CAI. Are there values that one must sacrifice to attain these levels of achievement? What are they? How can we protect these values in the schools as technology becomes increasingly prevalent in the next decades?

Hoban (1977) placed the highest priority on human values over technology:

> Explicity, the major theme is that a symbiotic relationship exists between educational technology and human values and that in this symbiosis, human values are or should be invariantly transcendent.

These positions may represent extremes among the many currently in the literature expressing concern about the relationship between human values and the role of technology. Though much has been written about this concern, less has been done to clarify what these concerns mean for classroom practice. In the decades immediately ahead, teachers and others responsible for classroom environments must address these complex questions. The answers such practitioners provide will do much to shape the nature of instruction in the 21st century.

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# Response

Robert B. Davis

How we use computers in education may well shape the future of education. Though it won't be the only influence, it is likely to be an important one. This is alarming because decisions about the educational uses of computers are not being made in the thoughtful, careful way that is called for. DeVault's excellent review is entirely correct (and stands virtually alone!) in looking at the pre-computer practices from which computer uses have grown. From the use of flash cards to teach "addition facts" like 3 + 2 = ?" there have grown computer programs that ask "3 + 2 = ?" From memorizing verbal definitions in pre-computer mathematics lessons, there have grown computer programs that ask one to type in (or select) verbal definitions.

In the case of our own Madison Project work, in pre-computer days we became convinced (from observing students) that verbal definitions don't work well with most children. They know a cat when they see one, but they cannot give you a verbal definition of "cat," and they can't learn from verbal definitions of this type. Hence, we developed the *paradigm teaching strategy* (Davis and others, 1978; Davis 1980a, 1980b, 1981) to give children experience with a mathematical concept without trying to use words to tell them about it. (Would you like to use words to tell children about an elephant if they had never seen one? Or would you rather take them to the zoo and show them an elephant?) We have created mathematics lessons for computer delivery built on this idea of paradigm teaching strategy: the computer program gives children experience with fractions, functions, and negative numbers, and does not try to introduce these ideas by purely verbal statements.

Now, the Madison Project approach here may be wise, or it may be foolish. In this short note one cannot argue the ultimate merits of any particular approach. But DeVault's point—a major one!—is that the Madison Project computer lessons grew out of the pre-computer Madison Project teaching practices. This is not peculiar to the Madison Project. The same situation exists for all computer-delivered lessons (so-called "courseware"). Whatever teacher or other specialist developed the lessons, he or she was building on pre-computer practices and expectations. I do not deplore this—on the contrary, it is inevitable, at least at first. But it means we must ask: how good were the pre-computer lessons from which the computer-administered versions have grown?

## Computers Sharpen Choices

There is a reason why this has suddenly become critical. Perhaps above all else, computers compel us to make commitments. When a student is practicing, say, factoring polynomials under the guidance of a sympathetic teacher, there can be so much going on—so many transactions between the participants—that it may be hard to say exactly what is happening. Sometimes for better, and sometimes for worse, subtlety and ambiguity rule the day. But put that same activity on a computer-administered lesson ("CAI mode"), and it quickly takes a more definite shape. On the computer, it becomes definitely drill, or definitely a game, or definitely a demanding lesson in reading comprehension, or definitely some other thing. Much of the ambiguity is gone, and we are faced with questions such as: Do we really *want* this much drill? Do we really *want* this many games? Should factoring polynomials be presented in a game-like atmosphere, anyhow? For that matter, should factoring polynomials be treated like drill? Before computers, these choices were less sharp. (One of my teachers in junior high argued that because of typewriters, spelling had become more important—one could not hide misspelling under a cloak of illegibility. Similarly, the definiteness of computer lessons precludes hiding uncertainty under a cloak of ambiguity and subtlety.)

## Bases for Decisions

How, then, is one to choose among different possibilities for computer CAI lessons? At least four methods must be considered: (a) use of paper-and-pencil tests (especially multiple-choice tests), (b) use of methods for revealing the performance and present status of individual students, (c) use of task-based interviews, (d) direct examination of the computer lessons themselves.

There is abundant evidence that method (a), despite its unfortunate appeal (and its resultant popularity) is in fact the least satisfactory. Multiple-choice tests appear to produce "hard data." This appearance is deceiving. Such tests produce *numbers*, but do not give adequate descriptions of how students are *thinking* about mathematical problems. Erlwanger (1973) found students who seemed, on test scores, to be making satisfactory progress, but for whom many mathematical symbols were meaningless—for example, students who had no idea of the size of decimals and fractions, who did not know whether .7 was larger than 6 or smaller than 1. Alderman and others (1979) confirmed this, finding students who believed that 3/10 was equal to 3.10. Alderman and others also found that 50 percent of the advantage of one curriculum over another in one comparison study was due to the specific format in which questions were posed. Change the format, but not the content, of the questions, and half the advantage of the curriculum disappeared. Porter (1980) and his colleagues, in a group of careful studies, found very little commonality between what was presented in textbooks, what was taught in class, and what was covered on the best-selling tests (at the level of fourth-grade mathematics). *We have not been testing what we have thought we were teaching*. Other fundamental reasons for fearing that test results can lead us in wrong directions are presented in Houts (1977), Tyler and White (1979), and elsewhere.

Indeed, my own main concern about computers is not about computers themselves. It is that test scores, because they are easily obtained and erroneously believed to be "scientific," will lead us into making incorrect choices, and thus into misapplying the promise of computers.

Measurements cannot, by their nature, resolve fundamental questions (see, for example, Kuhn, 1962). Results can always be interpreted in different ways, *if really fundamental uncertainties are involved*. Suppose we suspected that Curriculum A was sexist. Would we be satisfied by comparing test or questionnaire results of students in Curriculum A with those of students in a control group? Surely not; if no differences were found, there would remain the possibilities that our test or questionnaire was not sensitive enough, that the effect on students developed slowly and required a longer period of time to produce effects, or that the control curriculum itself was sexist. (This is not fanciful. Alderman and others found that students in a CAI curriculum had serious misconceptions about mathematics, *but so did the students in the control group*. One cannot defend ineffective curriculums by arguing that they are no worse than other ineffective curriculums!)

The large-scale introduction of computers into education is likely to be comparable to the large-scale introduction of gasoline-powered vehicles. Automobiles (which were not our only possible choice!) have facilitated suburban living (which can be pleasant) and thus contributed to the decline of our central cities (which had been based on proximity). They made the United States vulnerable to the political demands of the OPEC nations; they have contributed to our unfavorable balance of payments in international trade; they played a role in the destruction of the urban transit system in Los Angeles; and they have proved severely harmful to the environment. Had we based our early decisions about motor vehicles on an unthinking reliance on measurements, would we have measured the right things?

*In the crucial decisions concerning computers, there can be no substitute for careful analytical thought, especially thought about our fundamental goals and fundamental values. This kind of analysis is NOT presently taking place.*

The references listed below pursue further the problem of making wise decisions about the use of computers, and suggest a variety of alternative ways of using computers in education, some of which have not yet received the attention they deserve.

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# IX. Calculators

J. Fred Weaver

Rarely, if ever before, has there been as much exploration and investigation regarding a particular aspect of mathematics instruction, over so wide an educational range, within so short a period of time, as has been the case concerning use of the electronic calculator. In the second of her state-of-the-art reviews, Suydam (1979a) asserted that "Almost 100 studies on the effect of calculator use have been conducted during the past four or five years. This is more investigations than on almost any other topic or tool or technique for mathematics instruction during this century" (p. 3). Reports of calculator use in school settings continued to be released, more or less unabated, during 1979 and 1980. In this chapter attention is given only to studies at the precollege level, grades K-12 (although many postsecondary investigations have been conducted and reported).

## Delimitation by Exclusion

There are several things this chapter does *not* purport to be. It is not a comprehensive or definitive listing of research on calculator use in school settings. Such a listing would do no more than duplicate material found elsewhere (for example, Suydam, 1979b). This chapter will not summarize extensively and review critically any particular collection of investigations on calculator use in school settings. Such summaries and reviews also are readily available elsewhere (Suydam, 1979c, 1979d; Roberts, 1980). And this chapter makes no attempt to systematically and formally integrate or synthesize research on calculator use, whether by the commonly used voting method or by a more sophisticated meta-analysis.

What follows is a somewhat subjective distillation of the essence of consequential research findings to date on calculator use in school settings, the implications of such findings for classroom instruction, and some indication of research directions that need to be taken during the 1980s.

## A Diverse Domain

The domain of research pertaining to calculator use in school settings has *diversity* as one of its principal attributes. Consider these illustrations.

At one extreme we find reports of things that may be termed "informal explorations" or "feasibility studies," which were limited in one way or another. For instance: in one case a sample of only *three* pupils was involved; in another case the exploration time consisted of *two class periods;* and some published reports described or illustrated ways in which calculators were used, and identified certain findings or conclusions, but without any supporting objective evidence or data.

At the other extreme we find reports of *experimental investigations* that were substantial in scope: in some instances the treatments extended over an entire school year; and in one instance the sample involved pupils from 50 classes, grades two through six, from five Midwestern states.

Between the two extremes we find a broad spectrum of investigations that vary markedly: in quality; in the ways in which, and the extent to which, calculators were used; in the class environments in which calculators were used; in the nature and scope of content involved; and in the effects considered (cognitive and affective). Typically included in these investigations were the following:

- drill on basic multiplication facts having factors of 7, 8 or 9;
- development of the "concept and skill of long division" at the fifth-grade level;
- a potpourri of work with rational numbers (in common- and/or decimal-fraction form), with percents, and with ratios and/or proportions;
- work within algebraic and trigonometric contexts, with varying degrees of content coverage as in other instances that follow;
- classes in general mathematics and consumer mathematics;
- work with remedial and/or low-achieving students, including the mildly handicapped;
- specialized-content classes or courses such as business arithmetic and chemistry;
- work that focused on particular properties such as "doing/undoing," and on problem-solving processes and strategies;

- consideration of student performance associated with a particular calculator *type* (for example, RPN—reverse Polish notation); and
- surveys pertaining to opinions and practices regarding calculator use in school settings.

The preceding categories are illustrative rather than exhaustive or definitive. Suydam's (1979b) listing should be consulted for more extensive classifications and for explicit references.

## Freedom from Fear

There is no doubt that many calculator investigations have been prompted by a frequently expressed *fear* on the part of those persons who believe that "The principal objectives of mathematics instruction (at least in K-9) are that children learn the basic facts and pencil-and-paper algorithms. Such learning will not occur if hand-held calculators are made available in the schools" (Shumway, 1976, p. 572).[1]

Based on research evidence, is there valid cause for such fear (which is more intense the lower the grade level)? In seeking to answer that question I have drawn heavily on conclusions from several research reviews, each of which was fully cognizant of limitations inherent in certain of the investigations involved.

First, Suydam (1977) summarized 40 findings from 21 experimental and several action or preliminary investigations in which instructional effects of one kind or another were compared for *calculator* and *noncalculator* groups: "In 19 cases the Calculator group achieved significantly higher on pencil-and-paper tests (with which the calculator was not used). No significant differences were found in 18 instances. *In only three instances was achievement significantly higher for the Noncalculator group.*" Suydam concluded, "Such gross tabulations provide some support for the belief that calculators can be used to promote achievement" (p. 1, italics added).

Next, in a subsequent review involving a more extensive research base, Suydam (1978) indicated that "In most of the studies at the elementary school level, the data were collected to provide an answer (to parents and school boards, as well as to teachers) to the question, *'Will the use of calculators hurt mathematical achievement?'* The answer appears to be

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[1] This quotation was Shumway's way of summarizing the argument of those opposed to calculator use in schools. The statement should *not* be construed to reflect Shumway's own view, or mine, regarding "the principal objectives of mathematics instruction" and the role of calculator use in relation thereto.

*'No.'* . . . What we do know is that the calculator, in general, facilitates mathematical achievement across a wide variety of topics, and this finding is verified at both elementary and secondary levels" (p. 7, italics added).

And a year later, in discussing the investigations upon which her second state-of-the-art review was based, Suydam (1979a) stated, "Many of these studies had one goal: to ascertain whether or not the use of calculators would harm students' mathematical achievement. The answer continues to be 'No.' The calculator does not appear to affect achievement adversely. In all but a few instances, achievement scores are *as high or higher* when calculators are used for mathematics instruction (but not on tests) than when they are not used for instruction" (p. 3).

Roberts (1980) reached similar conclusions, along with some additional ones also cited by Suydam, in his critical review of 11 elementary- and 13 secondary-level investigations:

> The majority of the studies completed at the elementary level showed computational advantages (6 of the 11) from the introduction of calculator usage into the mathematics instruction, . . . However, in only one study of the five investigating concepts were there conceptual benefits due to calculator usage and in only one study of the four investigating attitudes were there attitudinal benefits (p. 76).
>
> A majority of the secondary-level studies (6 of the 11 computation studies) found computational benefits due to calculator use. However, as was the case in the elementary studies, very little support was found for the hypothesis that calculator benefits transfer to the more conceptual (1 of the 8 concept studies) and affective areas (2 of the 9 attitudinal studies) (pp. 79-80).
>
> There seems to be little doubt about the computational value associated with calculator use. . . . However, [with respect to] conceptual and attitudinal impacts due to calculator use, there is less consensus as to what facts can be gleaned from the research literature (p. 94).

Since the review by Roberts and those by Suydam, two additional comprehensive investigations regarding the effects of calculator use vs. nonuse at the elementary-school level have been reported.[2]

Findings from a year-long investigation involving two different instructional programs led Moser (1979) to conclude that "Use of calculators with ongoing curricula at the second- and third-grade levels had no harmful effect upon arithmetic achievement" (p. xiii).

Two reports of a study—funded by the National Science Foundation and conducted in grades two through six (50 classes from five Midwestern states, with treatments that were in effect for 67 school days within an 18-week period)—cited these observations:

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[2] These two additional investigations were included in Suydam's most recent state-of-the-art review (August 1980) which had not been released at the time this chapter was prepared.

"The results . . . show no evidence of a decline in mathematics learning in classes that used calculators and there was some evidence that children in the primary grades benefit from using calculators in the study of mathematics" (Wheatley and others, 1979, p. 21).

"Children grow significantly on basic fact and mathematics achievement tests taken without the use of calculators regardless of whether or not calculators were used during instruction. . . . Children . . . did not develop any of the feared debilitations when tested without calculators because of calculator use for instruction" (Shumway and others, 1981, pp. 139, 140).

What, then, can be concluded about the *fear* that calculators may inhibit mathematics learning in schools? All in all, when calculators were used in the variety of ways investigated to date across a rather wide range of grade levels and content areas, evidence suggests that we have *no cause for alarm or concern* about potentially harmful effects associated with calculator use. This is particularly true with respect to computational performance, for which a nontrivial amount of evidence of *facilitating* effects has been reported. Even in the case of conceptual and affective aspects of mathematical learning, there is no extensive or strong body of evidence that suggests any pronounced inhibitory effects associated with calculator use. Seldom is the research literature so clear as it is in this respect.

### An Implication for Classrooms at the Outset of the 1980s

I am convinced that we can embark on school mathematics instruction at the outset of the 80s with *freedom from fear*—freedom from fear that calculator use will have harmful or debilitating effects on students' mathematical achievement. Fear that calculator use will have marked negative cognitive or affective influences on students can no longer be used as a reason, or an *excuse*, for not welcoming and including calculators among the instructional aids and materials that have potential contributions to make in connection with school mathematics programs. The extent or fullness of that potential, however, remains to be ascertained.

## How Calculators Are Used

Several surveys have been conducted to find out how calculators are used by classroom teachers in connection with their mathematics instruction. Suydam (1978) indicated:

> At the elementary school level, four types of uses are predominant:
>
> (1) Checking computational work done with pencil and paper.
>
> (2) Games, which may or may not have much to do with furthering the mathematical content, but do provide motivation.
>
> (3) Calculation: when numbers are to be operated with, the calculator is used with the regular textbooks or program.
>
> (4) Exploratory activities, leading to the development of calculator-specific activities where the calculator is used to teach mathematical ideas.
>
> At the secondary school level, the emphasis varies:
>
> (1) Calculation, used whenever numbers must be operated with.
>
> (2) Recreations and games.
>
> (3) Exploration: because secondary school mathematics teachers' backgrounds are generally good, there is much more of this type of activity than at the elementary school level. In addition, the students who continue in higher-level courses are often intrigued to explore.
>
> (4) Use of calculator-specific materials. There is at least one text integrating the use of calculators, with several others being field-tested (p. 4).

And from their survey of calculator use in grades 1, 3, 5, and 7, Graeber and others (1977) reported, "In the first grade, calculators were used most frequently for drill; the next three most frequent usages were for checking, motivation, and remediation. Use of the calculator for drill decreased with grade level. Above first grade the most frequent usage was for checking. Motivation and word problems were the next most frequently reported uses for calculators at the higher grade levels" (quoted by Suydam, 1978, p. 5).

It really is not surprising to find that the most common uses of calculators are relatively pedestrian ones. This should change as teachers learn about and personally explore more significant roles for calculators.

## Looking Ahead

Although I can safely conclude that students' use of calculators will not inhibit their mathematical learning, *new research directions* and *imaginative curriculum development* are pre-eminent among things that are needed during the 1980s. The close relationship between them has been emphasized in a National Institute of Education and National Science Foundation (NIE/NSF) document (n.d.), *Report of the Conference on Needed Research and Development on Hand-Held Calculators in School Mathematics:* "Research must go hand in hand with development. . . . Developers should review relevant research in designing their curriculums, and researchers should investigate existing curriculum materials in choosing suitable contexts for their investigations" (p. 9).

In view of this reciprocating relationship between research and development, one is not necessarily distinguished from the other in some of the material that follows.

### A Broad Research Need

Begle's examination of reports of calculator investigations led him to conclude that "In almost all these studies, the calculator was used merely as a supplement to a regular course. We have yet to see the results of evaluation of instructional programs which explicitly make use of the special capabilities of calculators" (p. 114).

In a similar vein, Roberts (1980), for instance, contended that a "crucial" consideration of future research "will be the necessity to develop treatments that utilize unique capabilities inherent to calculators. So far, most studies have not adequately integrated calculator use into the instructional process" (p. 95).

### In Defense of the Past

I certainly concur with Begle and Roberts in their needs assessment. But it is important to recognize that calculator research to date has been essentially the first phase of an evolutionary process that is more or less natural, and not at all undesirable.

For one thing, some *feasibility* investigations—limited in scope, duration, and control—were necessary. We had to know whether certain things were even plausible—whether certain expectations were at all realistic—before more substantive studies could be considered at all sensibly. If young children, for instance, were prone to make many errors in using a calculator keyboard and in reading a calculator display (which we now know is *not* commonly the case), certain subsequent investigations involving young children would have been pointless. And knowledge of whether pupils are sufficiently sensitive to the use of various technical features of a calculator (such as automatic constants, memories of one kind or another, and logic systems involved) would be essential to deciding whether to even attempt to investigate certain calculator uses, treatments, or algorithms.

For another thing, it is not at all surprising or undesirable that many studies have been tied rather closely to existing curriculums. Marked changes in curriculums cannot be effected suddenly, desirable though some changes may be. If teachers and students were to use calculators at all, such use had to be first within the context of present curricular content. Moser (1979) was of the conviction that "research with an existing curriculum is judged to be a necessary prerequisite for future research" (p. 14).

The extensive suggestions he gave to second- and third-grade teachers in detailed day-by-day written form, and the ways in which researchers were available daily to consult with teachers and pupils in the "five-states study" (Wheatley and others, 1979; Shuway and others, 1981) represent exemplary efforts to effect *calculator-assisted* instruction *within existing curriculums* at the elementary-school level. In instances in which similar efforts were undertaken at middle- and secondary-school levels, work generally was of shorter durations of time with selected topics or pieces of content.

## Curricular Changes

If programs of school mathematics instruction (and research pertaining to it) are to take full advantage of the "special" and "unique capabilities" associated with calculators, some curricular changes—substantial ones, in certain respects—must be effected. One view of this at the pre-algebra level has been suggested by the National Advisory Committee on Mathematical Education (1975):

> The challenge to traditional instructional priorities is clear and present. . . . First, the elementary school curriculum will be restructured to include much earlier introduction and greater emphasis on decimal fractions, with corresponding delay and de-emphasis on common fraction notation and algorithms. . . .
>
> Second, while students will quickly discover decimals as they experiment with calculators, they will also encounter concepts and operations involving negative integers, exponents, square roots, scientific notation and large numbers—all commonly topics of junior high school instruction. . . .
>
> Third, arithmetic proficiency has commonly been assumed as an unavoidable prerequisite to conceptual study and application of mathematical ideas. This practice has condemned many low achieving students to a succession of general mathematics courses that begin with and seldom progress beyond drill in arithmetic skills. Providing these students with calculators has the potential to open a rich new supply of important mathematical ideas for these students—including probability, statistics, functions, graphs, and co-ordinate geometry—at the same time breaking down self-defeating negative attitudes acquired through years of arithmetic failure (pp. 41, 42).

Less marked changes at higher instructional levels have been proposed. For instance, the NIE/NSF document (n.d.) suggested that "Use of calculators will require less revision of some current courses, such as high school algebra, geometry, and elementary functions and analysis. In all these courses, however, some parts need to be revised to include more applications that exploit the full potential of calculators" (pp. 11-12). Jewell's (1979) textbook analyses led him to conclude that approximately one-half of the content of algebra, geometry, and elementary functions texts and one-eighth of an algebra-trigonometry text could be appropriate for meaningful calculator application.

These illustrations indicate that across grades K-12 high priority must be given to research and development efforts that will (1) generate and evaluate instructional programs that make explicit use of special capabilities of calculators, (2) generate and evaluate treatments that use unique capabilities inherent in calculators, and (3) adequately integrate calculator use into the instructional process so that the calculator's presumed potential to facilitate and enhance the teaching and learning of school mathematics may be suitably assessed. Such assessment should give particular attention to the development and acquisition of problem-solving skills, mathematical concepts, and algorithmic processes.[3] I share Moser's (1979) belief that "the full benefit of calculator use in schools will never be realized until existing curricula are modified or new ones are developed that take advantage of a calculator's distinct features" (p. 14).

## More than Mathematical Content

Research and development efforts associated with curricular change should involve more than content considerations per se. I wish to illustrate this by citing the possibility of a somewhat unconventional instructional sequence that would seriously challenge a position commonly held by a good many persons—a position expressed in the following way by Judd (1975):

> "Students must have a good background in manipulative math experiences before they can understand the inputs and outputs of the calculator. . . . Don't in short, put a calculator in the hands of a student before he . . . . understands the nature of the processes basic to arithmetic. Only after the students understand the meaning of the function they are performing should they be given a magic box to carry them to completion" (p. 48).

Now consider examples of the forms:

$$a + b = n \qquad a \times b = n$$
$$a - b = n \qquad a \div b = n$$

---

[3] In relation to both classroom instruction and research, calculators give rise to a massive set of issues and problems associated with *testing*—a veritable "can of worms" much too intertwined to be considered within the space limitations imposed upon this chapter. One inkling of this is glimpsed in the NACOME Report (1975): "Present standards of mathematical achievement will most certainly be invalidated in 'calculator classes.' An exploratory study in the Berkeley, California public schools [Kelley and Lansing, 1975] indicated that performance of low achieving junior high school students on the Comprehensive Tests of Basic Skills improved by 1.6 grade levels simply by permitting use of calculators" (p. 42).

where $a$ and $b$ are particular integers that are given, and $n$ is to be calculated. It is assumed that students have been introduced to the integer *concept* (and referents for it), but not to operations on integers. Very deliverately let the calculator play a "magic box" role. For each operation in turn, let students select particular integers $a$ and $b$ (with some possible teacher guidance or suggestion in order to sample the domain fully) and use calculators to generate corresponding sums, or differences, or products, or quotients;[4] and then use a particular set of assignments as the basis for intuiting the assignment or operational rule. Only *after* that would attention be directed to uses for and applications of an operation on integers and the calculation rule associated therewith—along with instructional activities that provide justification for a rule and its "sensibleness."

Such a procedure is not at all out of line with Wittrock's (1974) hypothesis which may be "succinctly, but abstractly stated, . . . that human learning with understanding is a generative process involving the construction of (a) organizational structures for storing and retrieving information, and (b) processes for relating new information to the stored information. Stated more directly, all learning that involves understanding is discovery learning" (p. 182).

It was recommended in the NIE/NSF (n.d.) document that due consideration be given to *"current psychological, behavioral, and learning theory models"* (p. 19). Not only is the suggested approach to operations on integers in keeping with that recommendation, but early in the last decade Fennema's (1972) research raised some serious questions about the commonly held belief that children's learning should invariably proceed from the concrete/manipulative to the abstract/symbolic. Her finding, that in second-graders' introductory work with whole-number multiplication certain advantages accrued from using a symbolic referent in contrast with a concrete referent, is of considerable potential significance and import. Calculators provide an excellent means of investigating this phenomenon further and in a wide variety of mathematical contexts.

And when such investigations also take into consideration work being done in areas such as information processing and cognitive psychology, in cognitive or learning styles, and in "right brain" vs. "left brain" functions, we very well may find that a concrete/manipulative-to-abstract/symbolic instructional sequence is not as sacrosanct as we seem to believe it to be. For some students at certain times it may be desirable, even preferable, to introduce new mathematical content in a symbolic rather than a concrete mode; and to change, even reverse, other facets of instructional sequences. During the 1980s our research and development efforts pertaining to calculators certainly should be planned to include specific tests of some of our cherished tenets regarding mathematics learning and teaching.

---

[4] The fact that the set of integers is not closed under division poses problems in the selection of values for $a$ and $b$ when working with examples of the form $a \div b = n$.

### Other "Time-ly" Issues

There is a sense in which the immediately preceding consideration involved a *time* factor—the time when certain things occur in an instructional sequence. I now wish to direct attention briefly to three other issues in which *time* is of consequence.

*First:* Although far from easy to implement and control, *longitudinal* studies of effects of calculator use are essential. It is far from sufficient to know what may, or may not, be expected from relatively short-term uses of calculators, even over an entire school year. Effects over time, over a period of several school years, must be assessed before we have sufficient knowledge to answer some of the questions that have arisen, and will arise, regarding the use of calculators. Often it is not simply time per se that is of consequence, but the *cumulative* effects over time. Cross-sectional investigations have a contribution to make, but they cannot give us precisely the same kind of information to be gleaned from longitudinal studies.

*Second:* We already know that calculators make it possible for students to work more exercises and to deal with more applications and problem situations (in which computations are necessary) within a given unit of time. But is "do more of the same thing," particularly in the case of exercises, using time to best advantage? Likely not, and we should assess the cumulative effects of introducing and extending additional content within the time "gained," as opposed to doing more of the same exercises or whatever. It is rather commonly believed that much is to be gained by expanding curricular content; but will that in reality be the case—and if so, in what way(s)?

As we seek to broaden the scope of curricular content (presumably through time to be gained by calculator use for routine computations), it is my hope that we will not repeat a serious mistake reflected in the report of the Cambridge Conference on School Mathematics (1963) in which some wholly unrealistic content expansions were suggested, especially for grades K-8. It is easy to become literally "too mathematical" for the rank and file of students within those grades, and it behooves us to be realistic and sensible regarding the development of a broader content base for calculator use and exploitation in grades K-8, and the planning of investigations related thereto.

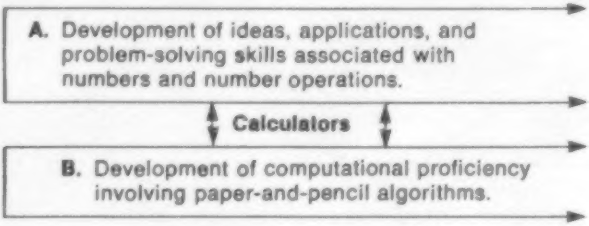
*Third:* As I suggested in an earlier source (Weaver, 1976), we need to investigate the long-term effects of (1) *introducing* certain pencil-and-paper algorithms *after* rather than before students have worked with numbers of certain magnitudes and domains using calculators, and (2) then emphasizing acquaintance and reasonable skill with such algorithms, rather than the attainment of a high degree of mastery coupled with a highly efficient level of performance.

Research and development activity should go even further and direct attention to the feasibility of an organizational pattern shown in Figure IX-1 for suitable phases of mathematics programs at elementary- and middle- or junior high-school levels.

A student's progress along each pathway is relatively independent of progress along the other. Calculators are used principally in connection with *A;* and when used in connection with *B,* their role is a different one than when used in connection with *A.* As computational proficiency is attained in an area within *B,* it may be used as needed and desired within *A.* But *lack of computational proficiency would never impede a student's progress in connection with A.*

The reorganized pattern suggested by Figure IX-1 rightfully makes *A* (rather than *B*) the principal focus of instruction.

**Figure IX-1. Calculator-Influenced Reorganized Instructional Pattern**


A. Development of ideas, applications, and problem-solving skills associated with numbers and number operations.
Calculators
B. Development of computational proficiency involving paper-and-pencil algorithms.

## Using Research-Based Information in the Classroom

I am confident that during the early years of the 1980s we will see evidence of much-needed efforts to implement the NIE/NSF (n.d.) recommendation that "New means should be explored to rapidly communicate results of experiments with calculators and proposals for their use to the teaching profession, especially at the elementary school level" (p. 20). But it is not sufficient simply to disseminate information, regardless of how widely and innovatively that may be done. It is essential that disseminated information be *used*, in one way or another, in connection with classroom instruction.

Within a broader context Suydam and Weaver (1975) made the following suggestions which are fully applicable within the present context of calculator-research concerns:

> Teachers should test research findings [and treatments] in their own classrooms. Remember that just because research says something was best for a *group* of teachers in a *variety* of classrooms, doesn't necessarily mean that it would be best for *you* as an individual teacher in your *particular* classroom. . . . Teachers have individual differences as well as pupils! . . .
>
> Teachers must be careful not to let prior judgments influence their willingness to try out and explore: open-mindedness is important. . . . Be willing to investigate. . . .
>
> Research is not an end in itself—it should lead to some kind of action. You decide to change, or not to change; you will accept something, you will reject something. . . . *Do* something as a result of research: incorporate the conclusions of research [as tempered by unique attributes of your own situation and circumstances] into your daily teaching (p. 6).

## In Conclusion

NCTM (1980) has recommended that "mathematics programs take full advantage of the power of calculators and computers at all grade levels" (p. 1), and has made the following related recommendations for *action:*

> *All students should have access to calculators and increasingly to computers throughout their school mathematics program.*
>
> Schools should provide calculators and computers for use in elementary and secondary school classrooms.
>
> Schools should provide budgets sufficient for calculator and computer maintenance and replacement costs.
>
> *The use of electronic tools such as calculators and computers should be integrated into the core mathematics curriculum.*
>
> Calculators should be available for appropriate use in all mathematics classrooms, and instructional objectives should include the ability to determine sensible and appropriate uses.
>
> Calculators and computers should be used in imaginative ways for exploring, discovering, and developing mathematical concepts and not merely for checking computational values or for drill and practice.
>
> *Curriculum materials that integrate and require the use of the calculator and computer in diverse and imaginative ways should be developed and made available.*
>
> Schools should insist that materials truly take full advantage of the immense and vastly diverse potential of the new media (p. 9).

Let us not only begin the 1980s with *freedom from fear*, but also progress through the 1980s with ever-increasing assurance of ways in which calculator use can facilitate school mathematics instruction, K-12, and enhance its quality. Forty-five years ago W. A. Brownell (1935) championed a change in school mathematics programs, formulating that which he termed the "meaning theory" in which "The basic tenet in the proposed instructional reorganization is to make arithmetic less a challenge to the pupil's memory and more a challenge to his intelligence" (p. 32).

Now, as we enter the 1980s, we are in a position to reformulate school mathematics programs in a manner that will free them from the shackles of the attainment of computational skills with pencil-and-paper algorithms as the basis upon which instruction is initiated, organized, and sequenced at the pre-secondary level; and will have analogous reorganizational implications for programs at the secondary level. *The calculator is the key.* Now is the time to turn that key in all earnestness.

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# Response

Richard J. Shumway

In order to facilitate the reader's interpretation of my remarks, I will attempt to summarize Weaver's major points as I see them:

1. The research on calculators and school mathematics is very *diverse* in scope and content.
2. We *need not fear* debilitating effects from student use of calculators.
3. We need to begin research efforts which explicitly study and make use of the *special capabilities* of calculators.
4. Could there be a *substantial impact* on school mathematics programs for student calculator use?
5. Can the calculator greatly *facilitate exploratory approaches* to mathematics?
6. Calculator research efforts should be quickly and effectively *communicated*.
7. Calculator research findings should be *tested by classroom* teachers in their own classrooms.
8. *All* students should have access to calculators.
9. "We are in a position to reformulate school mathematics programs in a manner that will free them from the shackles of the attainment of pencil-and-paper computational algorithms and skills as the basis upon which instruction is initiated, organized, and sequenced."

In spite of Weaver's warnings that he would not provide an extensive review of the research on calculators, I find his major points follow with scholarly care from current work and his own thoughtful experiences and deliberations. My own experiences and knowledge of the literature causes me to resonate fully with these points and add my wholehearted endorsement.

However, as a co-conspirator in the calculator revolution, I find Weaver strangely mute on several important points and school strategies for implementation of calculator use for mathematics instruction.

## Attitude

One of the most powerful and consistently reported effects of student use of calculators is the high enthusiasm and valuing students have for calculator-aided mathematics activities. Any device which causes so much pleasure to be associated with mathematics and increases the probability appropriate mathematics strategies will be chosen for problem solving deserves special note. I also believe parental openness to calculator use can be influenced significantly by the enthusiastic response children exhibit for calculator-aided mathematics. So in the pragmatic real-world problems of making calculators available to all children, such attitude results are most important.

## Computation

It would be well to note, obvious as it is, that calculators are the quickest, most accurate computational algorithms available to children today. In fact, the primary function of a calculator is to compute, and in the hands of children, the calculator serves the computational function better than any other technique or device in existence.

The effect of calculator use on measurement of mathematics achievement is important for two primary reasons. First, students' ability to perform computations is significantly improved through the use of a calculator. Secondly, few teachers see the logic of training students *with* calculators and then testing students and evaluating teachers *without* student use of calculators. For these reasons, I believe it is of the highest priority that all testing, classroom and standardized, should be done with student use of calculators immediately. Few teachers are going to make significant use of calculators, nor are tests and textbooks likely to be designed for calculator use until calculators are actually used for testing. Consequently, the first needed step is to use calculators for all testing. The inconveniences of such a plan are minor compared to the significant delays failure to take such action will cause.

In my view, mathematics curriculum is most properly influenced by three factors: mathematical structure, learning theories, and societal needs. Weaver asserts mathematical structure can be supported by calculator use, calls for more research on learning and calculators, but is unfortunately mute on societal needs. It would seem societal needs might provide significant support for the use of calculators by all children. A recent survey[1] of 100 "random" occupations reports that fully 98 percent of those persons involved used a calculator. How many company presidents would support a personnel manager's recommendation that all new employees spend 120 hours learning skills by hand which are currently done by machines in the plant? Is our current education program guilty of such an error today? The argument seems strong that, based on societal needs, calculators should be used for mathematics instruction.

## Communication

Weaver's points about the quick communication of results and the testing of the ideas with teachers' own students are excellent. No research ideas should be introduced without careful, local evaluation. Such evaluation can do wonders for parent, teacher, and administrator receptivity to the use of calculators.

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[1] Saunders, H. "When Are We Ever Gonna Have to Use This?" *Mathematics Teacher* 73 (1980): 7-16.

I was most excited by the quote near the end that Weaver cites admonishing one to "do something as a result of research." Sword in hand, I look for suggestions for what to do and all I find is "at the outset of the 1980's" I can embark on calculator use "with freedom from fear." In conclusion, however, Weaver supports the recommendations for action made by NCTM which suggest basically that all students should have access to calculators, they should be supported by budget, and integrated into the core mathematics curriculums. Weaver's citing of the NCTM Agenda for Action Recommendations is most appropriate and provides the best national endorsement for calculator use currently available. But here is what I wish he had said:

1. Calculators should be available for use at all times by all students.

2. All standardized testing should be done with calculators available. (Right now, do not wait for tests to be rewritten. Start with your school.)

3. Paper-and-pencil algorithms should only be taught to enhance and enrich mathematical thinking and not used or practiced as a computational tool.

4. Teachers should ask for, develop, and use supplementary materials that support students' ability to learn mathematics and solve problems. Dramatic changes ought to occur in mathematics curriculums. Material, trivially done with calculators, should be thrown out. The focus should be on activities that teach children mathematical thinking while using a calculator.

5. Programs should be evaluated carefully.

6. Expectations that children enjoy mathematics should be present.

Are these recommendations unsupported by research? Maybe; but as Weaver himself states, there is more research on calculators than on any other topic. Can researchers, teachers, and administrators not "do something" now? Assuming careful testing, are we not better prepared than ever before to take such a strong stand regarding the use of calculators? When I think of my own children, I want to say: "Don't waste their time doing trivia! Give them a calculator and get on to teaching them the mathematics they cannot now do." I was hoping Weaver would close with strong action items. Perhaps his quiet, reasoned approach is best, but I'm for a little dramatic action with careful testing.

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